Australian Mathematics Competition

2021 Questions and Solutions

A KEPERT & M CLAPPER

AUSTRALIAN MATHS TRUST
AUSTRALIAN MATHEMATICS COMPETITION

Published by

AMT PUBLISHING

Australian Maths Trust
170 Haydon Drive, Bruce ACT 2617
AUSTRALIA
Telephone: +61 2 6201 5136
www.amt.edu.au

AMTT Limited ACN 083 950 341

Australian Mathematics Competition Solutions ISSN 2652-7235

Contents

About the Australian Mathematics Competition

The Australian Mathematics Competition (AMC) was introduced in Australia in 1978 as the first Australia-wide mathematics competition for students. Since then it has served thousands of Australian secondary and primary schools, providing feedback and enrichment to schools and students. A truly international event, there are entries from more than 30 countries across Asia, the Pacific, Europe, Africa and the Middle East. As of 2021, the AMC has attracted more than 15.5 million entries.

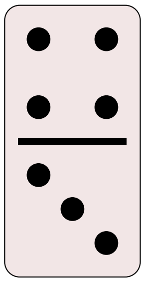
The AMC is for students of all standards. Students are asked to solve 30 problems in 60 minutes (Years 3–6) or 75 minutes (Years 7–12). The earliest problems are very easy. All students should be able to attempt them. The problems get progressively more difficult until the end, when they are challenging to the most gifted student. Students of all standards will make progress and find a point of challenge.

The AMC is a fun competition with many of the problems set in situations familiar to students and showing the relevance of mathematics in their everyday lives. The problems are also designed to stimulate discussion and can be used by teachers and students as springboards for investigation.

There are five papers: Middle Primary (Years 3–4), Upper Primary (Years 5–6), Junior (Years 7–8), Intermediate (Years 9–10) and Senior (Years 11–12). Questions 1–10 are worth 3 marks each, questions 11–20 are worth 4 marks, questions 21–25 are worth 5 marks, while questions 26–30 are valued at 6–10 marks, for a total of 135 marks.

Middle Primary Questions

1. How many dots are on this domino?

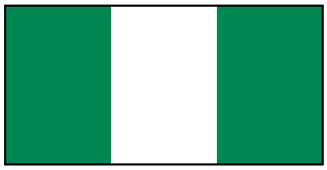


(A) 5 (B) 7 (C) 9 (D) 10 (E) 11

2. What is the difference between 14 and 2?

(A) 28 (B) 16 (C) 12 (D) 10 (E) 7

3. This Nigerian flag is white and green.
What fraction of it is green?



(A) one-third (B) one-quarter (C) one-half (D) two-fifths (E) two-thirds

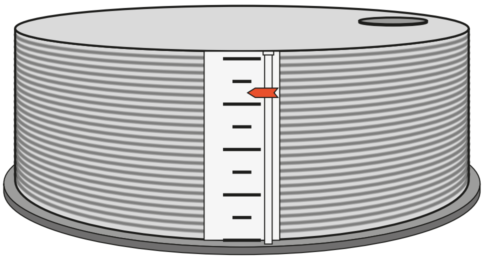
4. $234 + 100 =$

(A) 23400 (B) 1234 (C) 120304 (D) 334 (E) 244

5. How many minutes are in a quarter of an hour?

(A) 4 (B) 10 (C) 15 (D) 20 (E) 40

6. My tank can hold 80 kL of water.
The indicator on the tank shows the water level inside the tank.
Which of the following is closest to the amount of water in the tank?



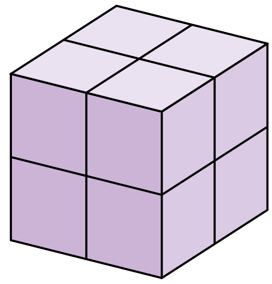
(A) 35 kL (B) 45 kL (C) 55 kL (D) 65 kL (E) 75 kL

7. Which number makes this number sentence true?

$$\square - 5 = 9$$

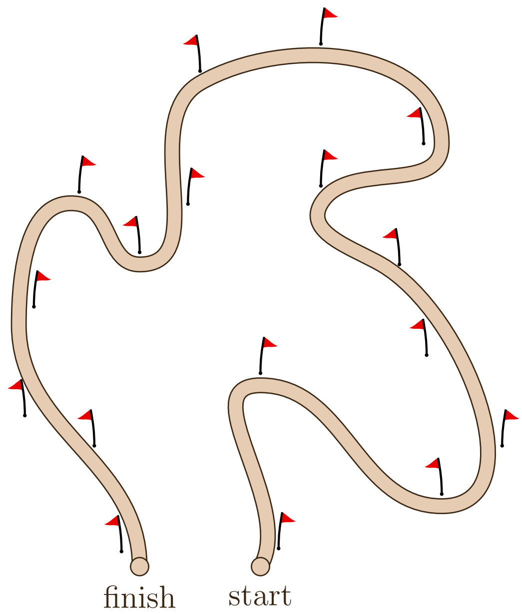
(A) 0 (B) 4 (C) 12 (D) 9 (E) 14

8. Each face of this cube is divided into 4 small squares. How many small squares are there on the outside of the cube altogether?



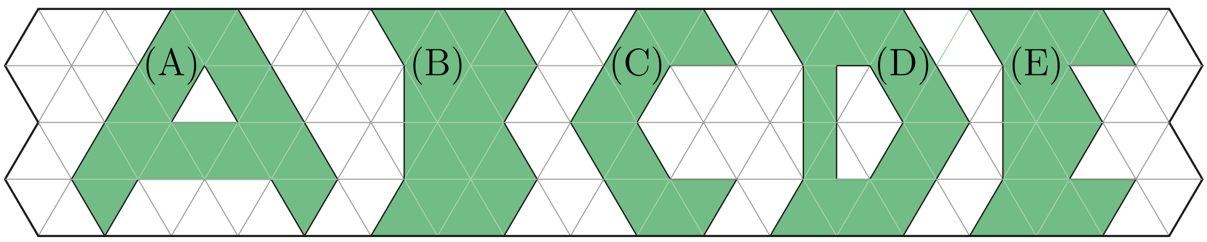
(A) 16 (B) 18 (C) 20 (D) 24 (E) 30

9. A cross country track is marked out with a number of flags as shown.
How many of the flags will be on the left of the runners when they pass them?



(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

10. Which one of these shaded areas is the largest?

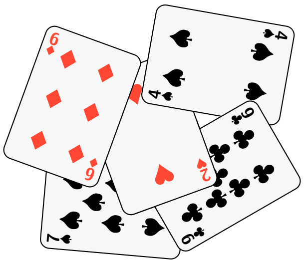


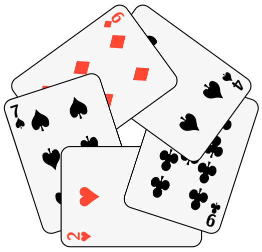
11. Leo is waiting in line at school. There are four students ahead of him and twice as many behind him. How many students are in this line?

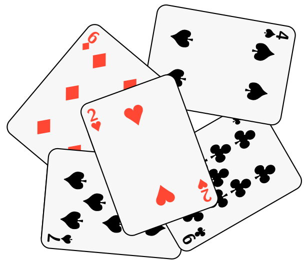
(A) 4 (B) 8 (C) 9 (D) 12 (E) 13

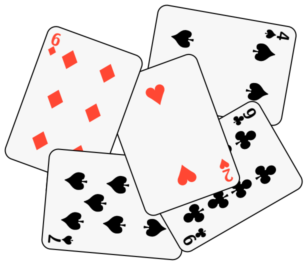
12. I am shuffling a deck of cards but I accidentally drop a card on the ground every now and then. After a while, I notice that I have dropped five cards.

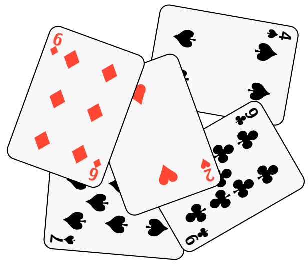
From above, the five cards look like one of the following pictures. Which picture could it be?

(A) 

(B) 

(C) 

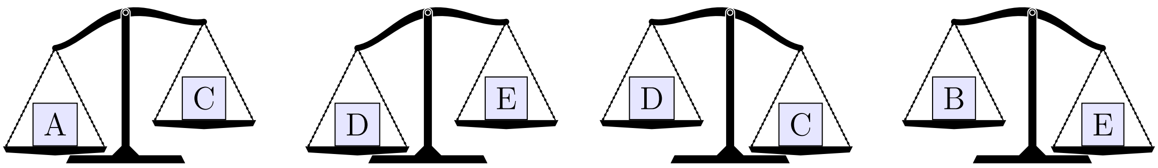
(D) 

(E) 

13. Kayla had six apples. She cut them all into quarters and shared them equally between her three brothers and herself. How many apples do they each receive?

(A) 1 (B) 3 (C) $1\frac{1}{4}$ (D) $1\frac{1}{3}$ (E) $1\frac{1}{2}$

14. Five boxes are compared on a balance.



Which of the five boxes is lightest?

(A) A (B) B (C) C (D) D (E) E

15. Lydia is saving for a cricket bat. The sports shop has the bat she wants for \$56 and her grandfather has promised to pay half the price.

She has saved \$16. How much more does she need to save before she can buy the bat?

(A) \$4 (B) \$12 (C) \$20 (D) \$28 (E) \$36

16. Five cards with digits $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{4}$ and $\boxed{9}$ are arranged to form the largest possible 5-digit even number. Which digit is in the tens place?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 9

17. Each letter in this grid stands for a number from 1 to 6.

The numbers outside the grid are the sums of the values of all the letters in each row or column.

For example, in the first column, the values of M, L, L and R add to 16.

What is the value of the letter L?

M	M	F	F	16
L	H	U	H	10
L	F	U	H	11
R	R	R	H	13
16	13	16	5	

(A) 1 (B) 2 (C) 3 (D) 5 (E) 6

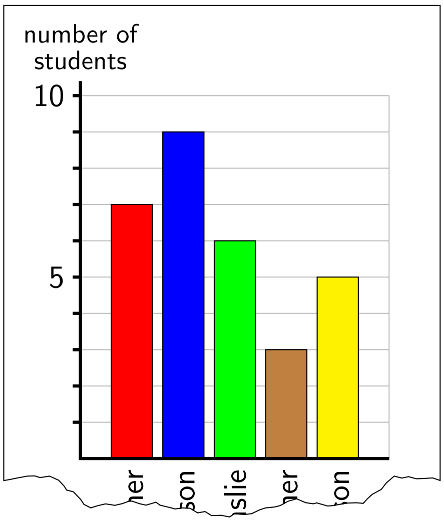
18. Greg is 19 years old, Karin is 26 and Anthony is 31. In how many years from now will their ages add to 100?

(A) 6 (B) 8 (C) 16 (D) 24 (E) 26

19. Mr Northrop's class has students from Ainslie, Turner, Downer, Watson and Dickson. He made a chart showing how many live in each suburb.

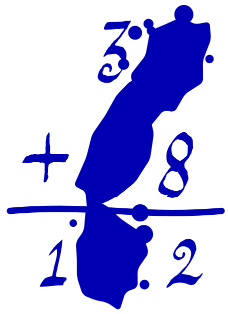
Unfortunately his dog tore the bottom of the chart, leaving only the last few letters of each suburb. He forgot the order of the suburbs on the chart, but he remembered that more students live in Downer than Watson.

How many students live in Turner?



(A) 3 (B) 5 (C) 6 (D) 7 (E) 9

20. Alexander's pen leaked on his addition homework, covering up three of the digits in the calculation shown. How many different possibilities are there for the correct working?



(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

21. Here are four sentences and their translations into Windarian, an invented language. The two lists are not in the same order.

English
Mum likes apples.
Dad likes oranges.
Brother loves apples.
Sister loves apples.

Windarian
Ato bem kito.
Awe tum kete.
Eke bem kete.
Alo tum kete.

How should we translate the sentence 'Mum loves oranges'?

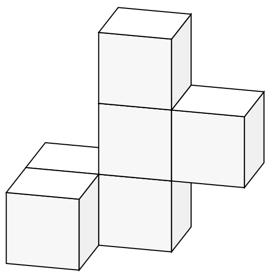
(A) Awe tum kete (B) Ato bem kito (C) Eke tum kito (D) Awe bem kete (E) Eke bem kito

22. The biscuit section in a cookbook has 6 pages. The sum of all the page numbers in this section is 147. What is the number of the last page in this section of the book?

(A) 26 (B) 27 (C) 28 (D) 29 (E) 30

23. Six white cubes are joined together as shown. The model is then painted blue all over.

When the model is pulled apart, how many faces of these cubes are still white?

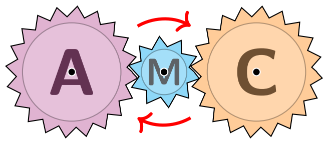


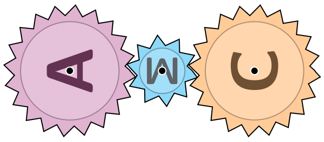
(A) 4 (B) 5 (C) 8 (D) 10 (E) 13

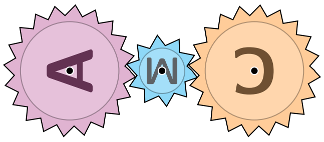
24. Three gears are connected as shown. The two larger gears have 20 teeth each and the smaller gear has 10 teeth.

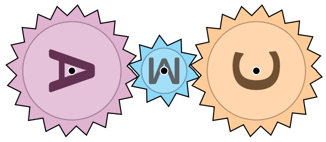
The middle gear is rotated half a turn in the direction of the arrows, turning the M upside down.

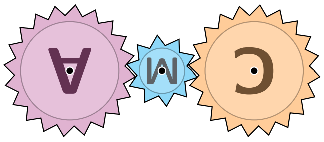
What do the three gears look like after this rotation?

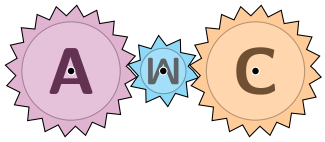


(A) 

(B) 

(C) 

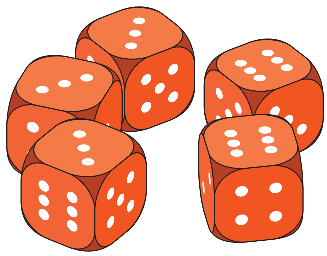
(D) 

(E) 

25. In a dice game, Yasmin rolls 5 standard dice, all at once.

She needs to roll a *full house*, which has a triple of one number and a pair of a different number.

How many different full house rolls are possible?



(A) 2 (B) 5 (C) 18 (D) 25 (E) 30

26. This is a magic square, so that all rows, columns and diagonals add up to the same sum.

Some numbers are already filled in.

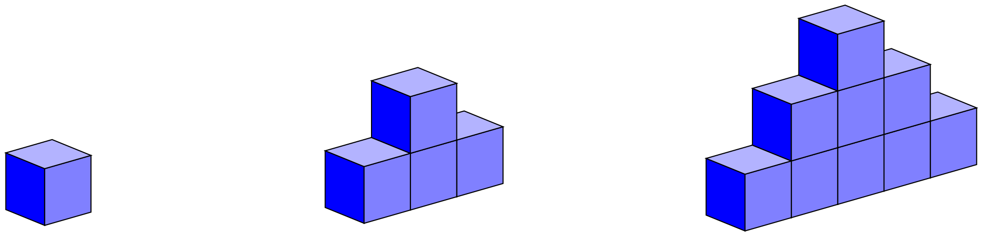
When we complete it and multiply the numbers in the three shaded squares, what do we get?

16	a	2	
	10	c	8
	b	7	12
4	15		1

27. Hayden saved \$1420 and Mitchell saved \$505. After they each spent an equal amount of money, Hayden had 4 times as much money as Mitchell. In dollars, how much did each of them spend?

28. The block pattern below has 1 block in the first tower, 4 blocks in the second tower, 9 blocks in the third tower and so on.

How many blocks are needed to make all of the first ten towers in this pattern?



First tower Second tower Third tower

29. Verity has 6 cards with digits 1, 2, 3, 4, 5 and 6.

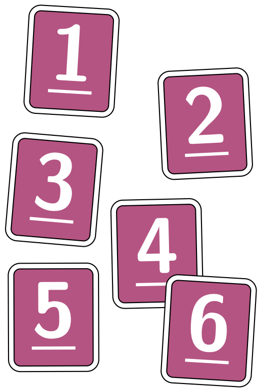
She arranges them to form three 2-digit numbers.

Only her first number is a multiple of 4.

Only her second number is a multiple of 5.

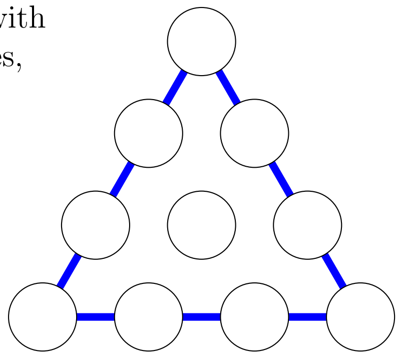
Only her third number is a multiple of 6.

What is the answer when she multiplies her first two numbers and then adds her third number?



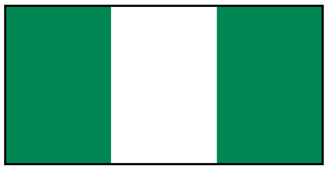
30. I want to place the numbers 1 to 10 in this diagram, with one number in each circle. On each of the three sides, the four numbers add to a *side total*, and the three side totals are all the same.

What is the smallest number that this side total could be?



Upper Primary Questions

1. This Nigerian flag is white and green.
What fraction of it is green?



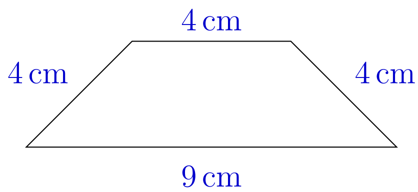
(A) one-third (B) one-quarter (C) one-half (D) two-fifths (E) two-thirds

2. Which number makes this number sentence true?

$$\square - 5 = 9$$

(A) 0 (B) 4 (C) 12 (D) 9 (E) 14

3. What is the perimeter of the quadrilateral shown?



(A) 13 cm (B) 15 cm (C) 17 cm (D) 19 cm (E) 21 cm

4. Which of the following decimal numbers has the smallest value?

(A) 0.0002 (B) 0.002 (C) 0.02 (D) 0.2 (E) 2.0

5. $\frac{1}{2} + \frac{2}{4} - \frac{4}{8} =$

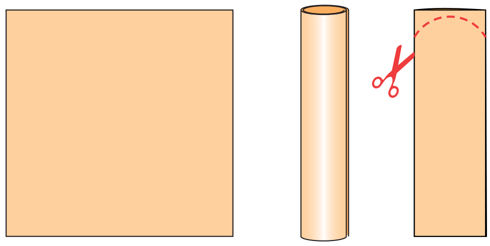
(A) $\frac{1}{2}$ (B) 1 (C) $1\frac{1}{2}$ (D) 2 (E) 4

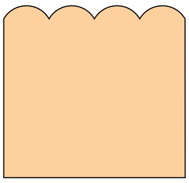
6. Suri has a number of 20-cent and 50-cent coins. Which of the following amounts of money is it **NOT** possible for her to make?

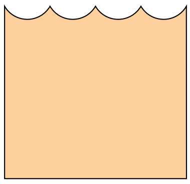
(A) 50 cents (B) 60 cents (C) 80 cents (D) 30 cents (E) 70 cents

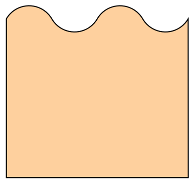
7. A square of paper is rolled up, pressed flat, and then cut as shown.

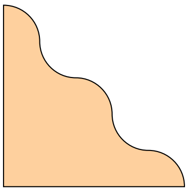
What could the sheet of paper look like when unrolled and laid flat?

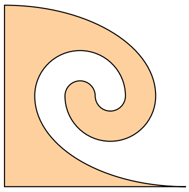


(A) 

(B) 

(C) 

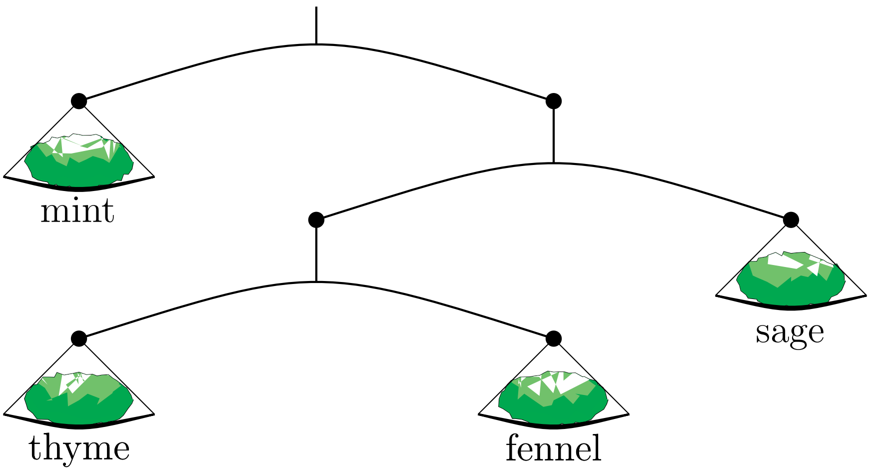
(D) 

(E) 

8. Leo is waiting in line at school. There are four students ahead of him and twice as many behind him. How many students are in this line?

(A) 4 (B) 8 (C) 9 (D) 12 (E) 13

9. Cassandra makes a healing potion from a mixture of herbs. She uses this balance to weigh out the herbs. If she uses 5 grams of fennel, how many grams of mint will she need?



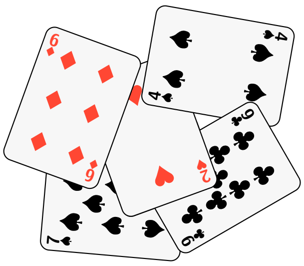
(A) 5 (B) 10 (C) 15 (D) 20 (E) 40

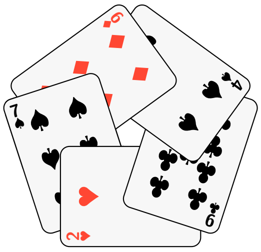
10. There are 14 pieces of fruit in a bowl. There are twice as many nectarines as pears, and half as many nectarines as apples. There are no other types of fruit. How many apples are there?

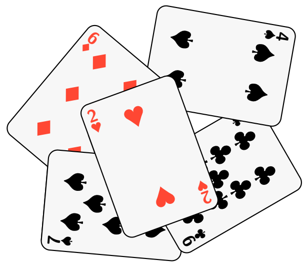
(A) 2 (B) 4 (C) 6 (D) 8 (E) 10

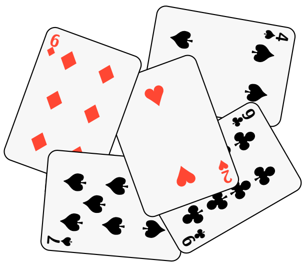
11. I am shuffling a deck of cards but I accidentally drop a card on the ground every now and then. After a while, I notice that I have dropped five cards.

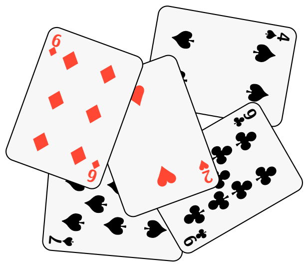
From above, the five cards look like one of the following pictures. Which picture could it be?

(A)


(B)


(C)


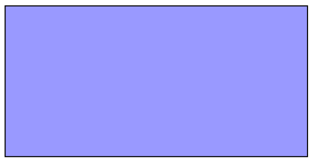
(D)


(E)


12. This rectangle has been made by joining two squares together.

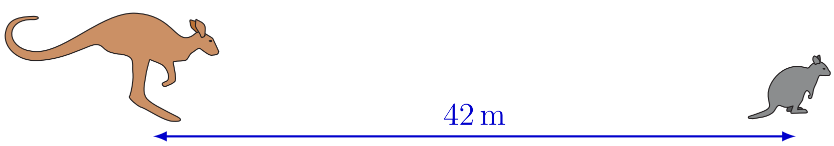
Each square has an area of 25 cm^2.

What is the perimeter of the rectangle?



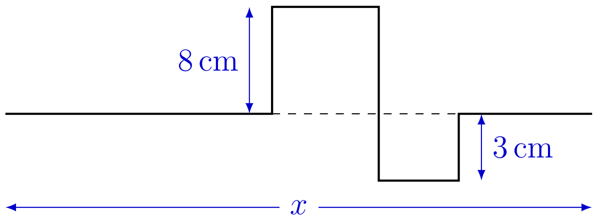
(A) 18 cm (B) 20 cm (C) 26 cm (D) 30 cm (E) 50 cm

13. A kangaroo is chasing a wallaby that is 42 metres ahead. For every 4-metre hop the kangaroo makes, the wallaby makes a 1-metre hop. How many hops will the kangaroo have to make to catch up with the wallaby?



(A) 8 (B) 10 (C) 11 (D) 14 (E) 21

14. A piece of straight wire is 50 cm long. Six right-angled bends are made in the wire, so that it ends up looking like the diagram shown:



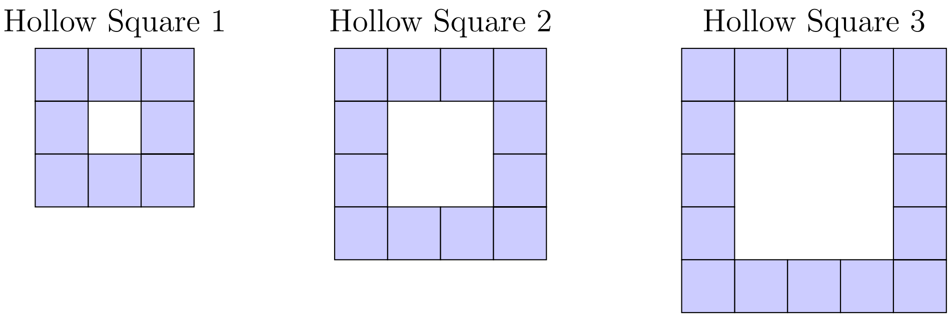
The lengths of two sections are shown. What is the length marked x?

(A) 28 cm (B) 31 cm (C) 34 cm (D) 36 cm (E) 39 cm

15. Margie and Rosie both live near Lawson train station. Each plans to catch the 10 am train. Margie thinks her watch is 10 minutes fast, but in fact it is 10 minutes slow. Rosie thinks her watch is 10 minutes slow, but in fact it is 5 minutes fast. Each of them leaves home to catch the train without having to wait on the platform. Who misses the train, and by how much?

(A) Margie by 10 minutes
(B) Margie by 20 minutes
(C) Rosie by 5 minutes
(D) Rosie by 15 minutes
(E) Neither of them

16. Sally was playing with block patterns and came up with this one she called *Hollow Squares*. They all follow the same pattern.



How many blocks would she need to make Hollow Square 7?

(A) 28 (B) 30 (C) 32 (D) 34 (E) 53

17. I have a jug containing 100 mL of liquid, which is half vinegar and half olive oil. How much vinegar must I add to make a mixture which is one-third olive oil?

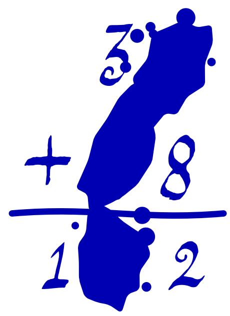
(A) 30 mL (B) 40 mL (C) 50 mL (D) 60 mL (E) 100 mL

18. It is 10 am now. What time will it be in 2021 hours time?

(A) 11 am (B) 1 pm (C) 3 pm (D) 4 pm (E) 5 pm

19. Alexander's pen leaked on his addition homework, covering up three of the digits in the calculation shown. How many different possibilities are there for the correct working?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6



20. Our school is organising a quiz night. They are expecting from 25 to 35 people to come. The people will be arranged in teams of 6 to 8 people.

What is the range of possible numbers of teams to expect?

(A) 4 to 5 (B) 4 to 6 (C) 5 to 6 (D) 3 to 6 (E) 3 to 5

21. In an election for school captain, there were 4 candidates and 453 students each voted for one candidate. The winner's margins over the other candidates were 31, 25 and 19.

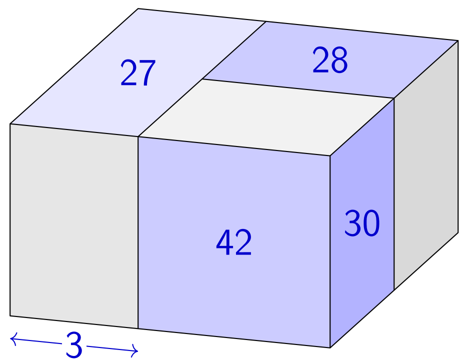
How many votes did the winner receive?

(A) 113 (B) 127 (C) 129 (D) 131 (E) 132

22. Three blocks with rectangular faces are placed together to form a larger rectangular prism.

All blocks have side lengths which are whole numbers of centimetres. The areas of some of the faces are shown, as is the length of one edge.

In cubic centimetres, what is the volume of the combined prism?

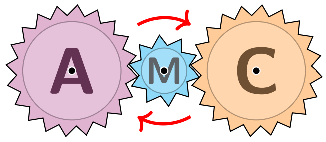


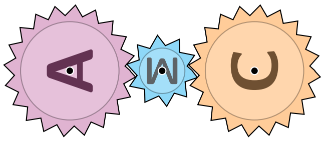
(A) 360 (B) 540 (C) 600 (D) 720 (E) 900

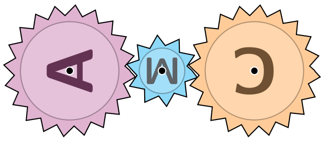
23. Three gears are connected as shown. The two larger gears have 20 teeth each and the smaller gear has 10 teeth.

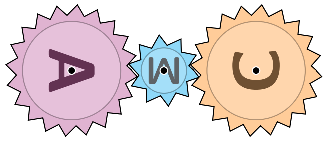
The middle gear is rotated half a turn in the direction of the arrows, turning the M upside down.

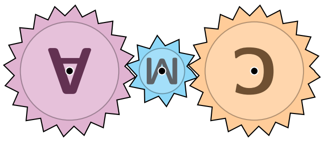
What do the three gears look like after this rotation?

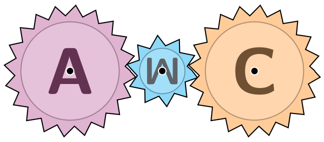


(A) 

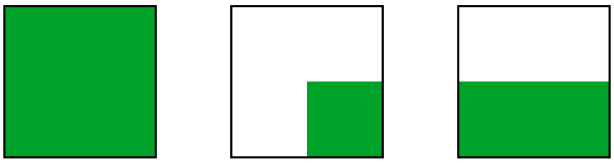
(B) 

(C) 

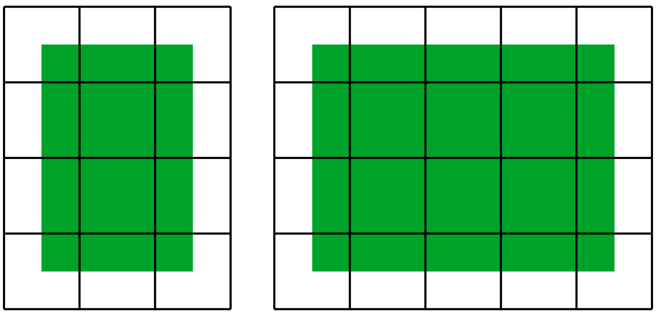
(D) 

(E) 

24. Anna has a large number of tiles of three types:



She wants to build a green rectangle with a white frame similar to those below.



She builds such a rectangle using as many tiles as possible while using exactly 20 completely green tiles. How many tiles will she use altogether?

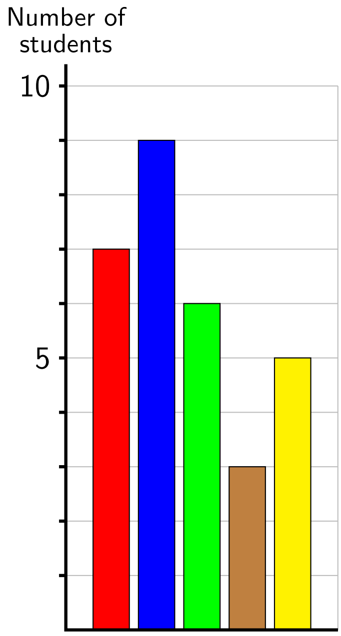
(A) 80 (B) 66 (C) 48 (D) 42 (E) 39

25. In Jeremy's hometown of Windar, people live in either North, East, South, West or Central Windar.

Jeremy is putting together a chart showing where the students in his class live, but unfortunately his dog chewed his survey results before he managed to label the five columns.

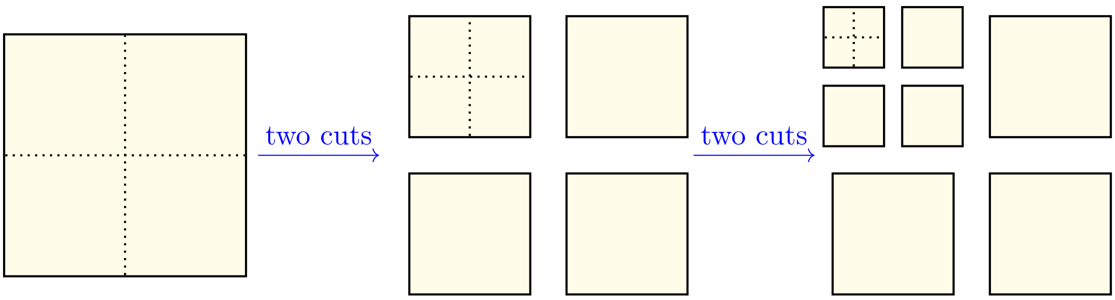
He only remembers two things about the survey: South Windar is more common than both East and Central Windar, and the number of students in North and Central Windar combined is the same as the total of the other three regions.

Using only this information, how many columns can Jeremy correctly label with 100% certainty?



(A) 0 (B) 1 (C) 2 (D) 3 (E) 5

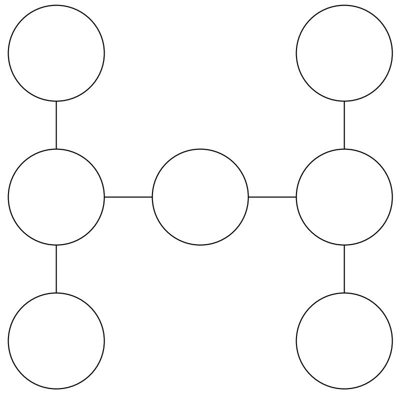
26. Pip starts with a large square sheet of paper and makes two straight cuts to form four smaller squares. She then takes one of these smaller squares and makes two more straight cuts to make four even smaller ones, as shown.



Continuing in this way, how many cuts does Pip need to make to get a total of 1000 squares of various sizes?

27. Seven of the numbers from 1 to 9 are placed in the circles in the diagram in such a way that the products of the numbers in each vertical or horizontal line are the same.

What is this product?

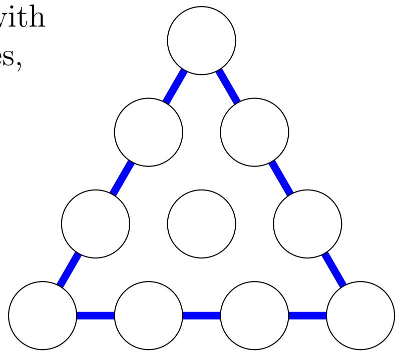


28. A hare and a tortoise compete in a 10 km race. The hare runs at 30 km/h and the tortoise walks at 3 km/h. Unfortunately, at the start, the hare started running in the opposite direction. After some time, it realised its mistake and turned round, catching the tortoise at the halfway mark.

For how many minutes did the hare run in the wrong direction?

29. I want to place the numbers 1 to 10 in this diagram, with one number in each circle. On each of the three sides, the four numbers add to a *side total*, and the three side totals are all the same.

What is the smallest number that this side total could be?



30. The sum of two numbers is 11.63. When adding the numbers together, Oliver accidentally shifted the decimal point in one of the numbers one position to the left. Oliver got an answer of 5.87 instead.

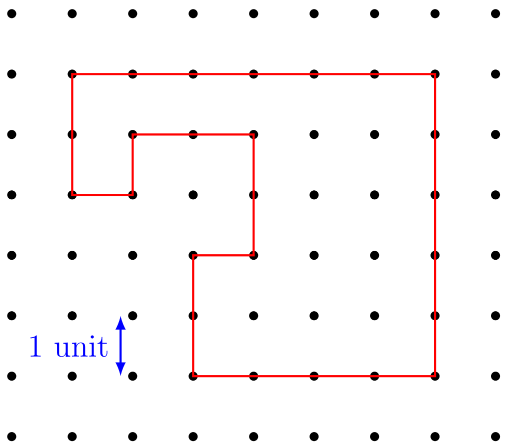
What is one hundred times the difference between the two original numbers?

Junior Questions

1. $2021 - 1202 =$

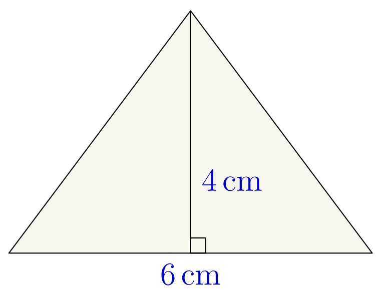
(A) 719 (B) 723 (C) 819 (D) 823 (E) 3223

2. What is the perimeter of this figure?



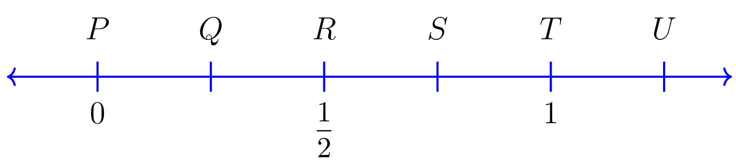
(A) 28 units
(B) 26 units
(C) 24 units
(D) 20 units
(E) 21 units

3. The area of this triangle is



(A) $10\,\text{cm}^2$ (B) $12\,\text{cm}^2$ (C) $12.5\,\text{cm}^2$ (D) $15\,\text{cm}^2$ (E) $16\,\text{cm}^2$

4. On the number line below, the fraction $\frac{3}{8}$ lies between

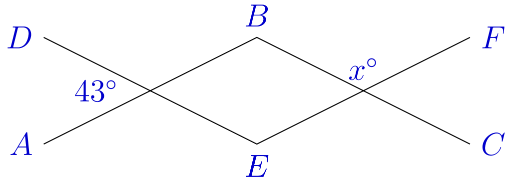


(A) P and Q (B) Q and R (C) R and S (D) S and T (E) T and U

5. Which of the following is closest to 2021?

(A) 202×100 (B) 22×1000 (C) 20.2×100 (D) 10×20.2 (E) 100×2.2

6. In the diagram, AB is parallel to EF and DE is parallel to BC. What is the value of x?



(A) 43 (B) 47 (C) 133 (D) 135 (E) 137

7. Mister Meow attempted the calculation $5 \times 2 + 4$, but accidentally swapped the multiplication and addition symbols. His answer was

(A) too low by 2 (B) too low by 1 (C) still correct (D) too high by 1 (E) too high by 2

8. Dad puts a cake in the oven at 11:49 am. The recipe says to bake it for 75 minutes. When should the cake come out of the oven?

(A) 1:04 pm (B) 12:34 pm (C) 1:54 pm (D) 1:19 pm (E) 12:04 pm

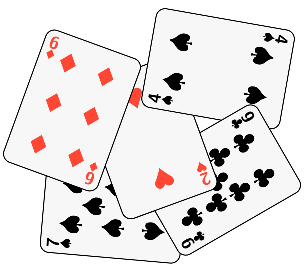
9. Damon made up a joke and sent it as a text message to three people in his class. These three each sent it to three other people in the class. No-one receiving the joke had seen it before. Including Damon, how many people now know the joke?

(A) 9 (B) 11 (C) 13 (D) 15 (E) 16

10. I am shuffling a deck of cards but I accidentally drop a card on the ground every now and then. After a while, I notice that I have dropped five cards.

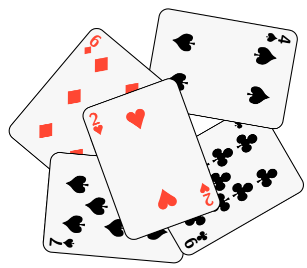
From above, the five cards look like one of the following pictures. Which picture could it be?

(A)

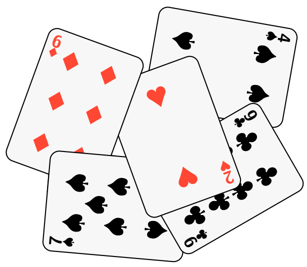


(B)

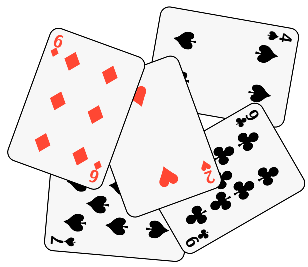
(C)



(D)



(E)

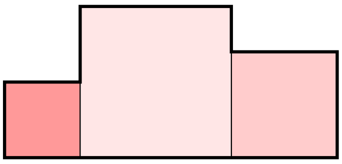


11. To feed a horse, Kim mixes three bags of oats with one bag containing 20% lucerne and 80% oats. If all the bags have the same volume, what percentage of the combined feed mixture is lucerne?

(A) 3 (B) 5 (C) 6 (D) 20 (E) 60

12. Three squares with perimeters 12 cm, 20 cm and 16 cm are joined as shown. What is the perimeter of the shape formed?

(A) 34 cm (B) 40 cm (C) 41 cm (D) 42 cm (E) 48 cm



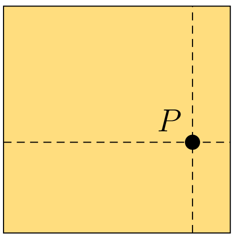
13. The odometer in my car measures the total distance travelled. At the moment, it reads 199 786 kilometres. I'm interested in when the odometer reading is a palindrome, so that it reads the same backwards as forwards. How many more kilometres of travel will this take?

(A) 25 (B) 125 (C) 15 (D) 205 (E) 2005

14. A square has an internal point P such that the perpendicular distances from P to the four sides are 1 cm, 2 cm, 3 cm, and 4 cm.

How many **other** internal points of the square have this property?

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9



15. How many different positive whole numbers can replace the $\triangle$ to make this a true statement?

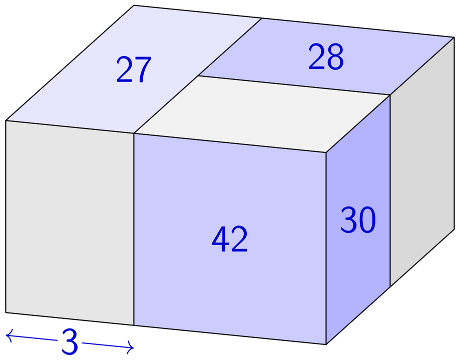
$$\frac{\triangle}{10} + \frac{1}{3} < 1$$

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

16. Three blocks with rectangular faces are placed together to form a larger rectangular prism.

All blocks have side lengths which are whole numbers of centimetres. The areas of some of the faces are shown, as is the length of one edge.

In cubic centimetres, what is the volume of the combined prism?



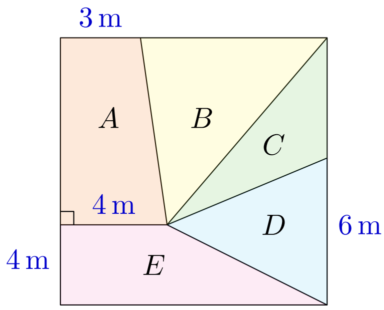
(A) 360 (B) 540 (C) 600 (D) 720 (E) 900

17. I have four consecutive odd numbers. The largest is one less than twice the smallest. Which of the following is the largest of the four numbers?

(A) 9 (B) 11 (C) 13 (D) 15 (E) 21

18. This is a square with sides of 10 metres.

From the constructions shown, which of the areas is the largest?



(A) A (B) B (C) C (D) D (E) E

19. Sandy, Rachel and Thandie collect toy cars. Altogether they have 300 cars.

Rachel has grown up and decides to give her cars away. If she gives them all to Sandy, then Sandy will have 180. If she gives them all to Thandie, then Thandie will have 200.

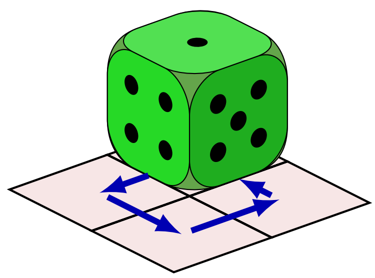
How many cars does Rachel have?

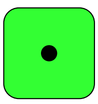
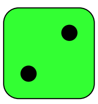
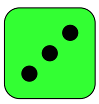
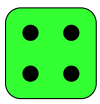
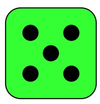
(A) 80 (B) 90 (C) 100 (D) 110 (E) 120

20. A standard dice numbered 1 to 6 with opposite sides adding to 7 is placed on a 2 by 2 square as shown.

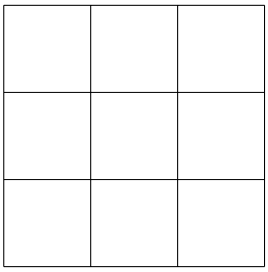
The dice is rolled over one edge onto each of the four base squares in turn and then back on to the original square, as indicated by the arrows.

Which side of the dice is now facing upwards?



(A)  (B)  (C)  (D)  (E) 

21. Leonhard is designing a puzzle for Katharina. It has nine squares in a 3×3 grid and a number of *clues*. Each clue is a number 1, 2 or 3 placed in one of the squares.



Katharina then has to find a solution by placing 1, 2 or 3 in each of the remaining squares so that no row or column has a repeated number.

What is the smallest number of clues that Leonhard could include so that his puzzle has exactly one solution?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

22. Grandma and Grandpa took their three grandchildren to the cinema. They purchased 5 seats in a row. Each grandparent wanted to sit next to two of the grandchildren.

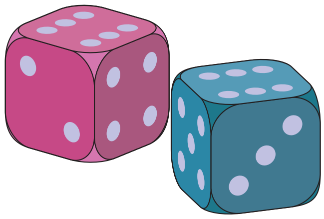
How many such seating arrangements are possible?

(A) 8 (B) 12 (C) 30 (D) 3 (E) 60

23. I have a 4 by 4 by 4 cube made up from 64 unit cubes. I paint 3 faces of the larger cube. Then I pull the cube apart. Which of the following could be the number of unit cubes with no paint on them?

(A) 16 (B) 21 (C) 24 (D) 28 (E) 36

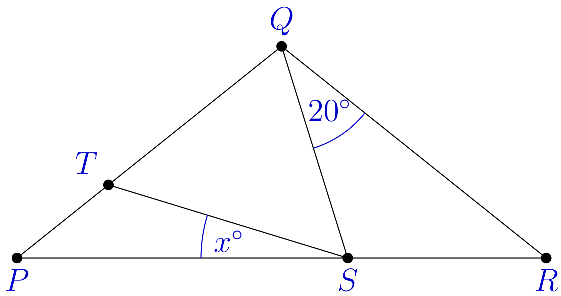
24. Ben and Jerry each roll a standard dice. If Ben rolls higher than Jerry, he wins; otherwise Jerry wins.



What is the probability that Ben wins?

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{5}{12}$ (D) $\frac{17}{36}$ (E) $\frac{1}{2}$

25. In the diagram, $\triangle PQR$ is isosceles, with $PQ = QR$. S is a point on PR and T is a point on PQ such that $QT = QS$, and $\angle SQR = 20°$.



The size of $\angle TSP$, in degrees, is

(A) 10 (B) 12 (C) 15 (D) 20 (E) 24

26. Starting with a 43×47 rectangle of paper, Sadako cuts the paper to remove the largest square possible.

With the remaining rectangle, she again cuts it to remove the largest square possible. She continues doing this until the remaining piece is a square.

What is the total perimeter of all the squares Sadako has at the end?

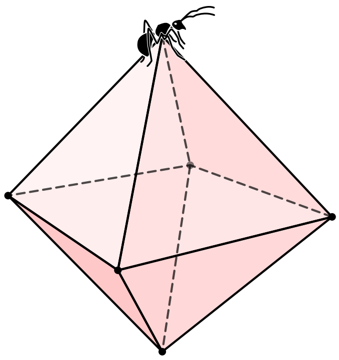
27. There are 14 chairs equally spaced around a circular table, and numbered from 1 up to 14. How many ways are there to choose two chairs that are not opposite each other?

28. A swimming medley consists of 100 metres of each of butterfly, backstroke, breaststroke and freestyle, in that order. I swim freestyle 3 times faster than breaststroke, and butterfly twice as fast as breaststroke, and my backstroke is half as fast as my freestyle. It takes me 6 minutes to swim the full medley. To the nearest metre, how far will I have swum after 4 minutes?

29. An ant's walk starts at the apex of a regular octahedron as shown.

It walks along five edges, never retracing its path. It visits each of the other five vertices exactly once.

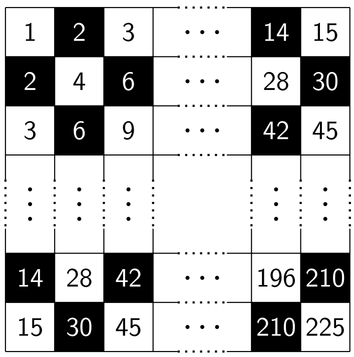
In how many different ways can the ant do this?



30. Consider a 15×15 grid of unit squares. In the square in row a and column b, we write the number $a \times b$.

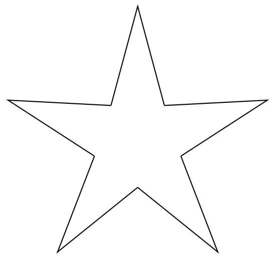
We then colour the squares black and white in a checkerboard fashion, so that the square labelled 225 is coloured white. The diagram shows the parts of the grid near each corner.

What are the last three digits of the sum of the numbers in the white squares?



Intermediate Questions

1. Each edge of this star is 2 cm long.
What is its perimeter?

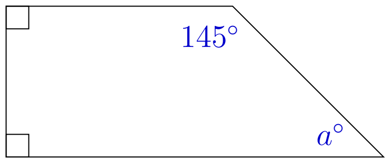


(A) 5 cm (B) 10 cm (C) 15 cm (D) 20 cm (E) 25 cm

2. The value of $2000 - 200 + 20 - 2$ is

(A) 1778 (B) 1782 (C) 1818 (D) 1822 (E) 1888

3. What is the value of a in the diagram?



(A) 35 (B) 45 (C) 55 (D) 65 (E) 75

4. What is 50% more than $\frac{1}{2}$?

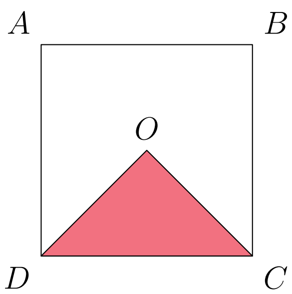
(A) $\frac{1}{4}$ (B) $\frac{5}{8}$ (C) $\frac{3}{2}$ (D) $\frac{3}{4}$ (E) 50.5

5.

$$\frac{1+3+5+7+9}{2+4+6+8+10} =$$

(A) $\frac{1}{2}$ (B) $\frac{5}{6}$ (C) $\frac{11}{12}$ (D) $\frac{9}{10}$ (E) $\frac{63}{256}$

6. Square $ABCD$ has centre O.
The shaded area is 16 square units.
What is the length of the side of the square?



(A) 4 (B) 8 (C) 16 (D) 32 (E) 64

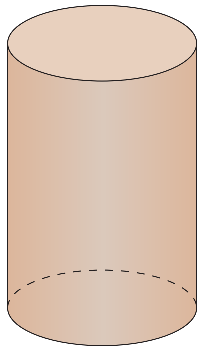
7. On the number line, which number is halfway between 10^2 and 10^4?

(A) 500 (B) 550 (C) 1010 (D) 2021 (E) 5050

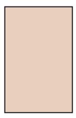
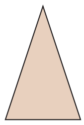
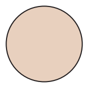
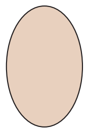
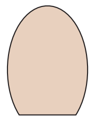
8. To feed a horse, Kim mixes three bags of oats with one bag containing 20% lucerne and 80% oats. If all the bags have the same volume, what percentage of the combined feed mixture is lucerne?

(A) 3 (B) 5 (C) 6 (D) 20 (E) 60

9. I have a solid block of wood in the shape of a cylinder. The top and bottom faces meet the curved side at right angles. Suppose that I slice the cylinder along a plane to create two smaller blocks of wood.



Which of the following could **not** be the shape of the resulting faces created by the slice?

(A)  (B)  (C)  (D)  (E) 

10. Diya timed herself cycling laps around her suburb. After five laps, her stopwatch indicated a time of 18 minutes and 15 seconds.

What was Diya's average time per lap?

(A) 3 minutes and 3 seconds (B) 3 minutes and 15 seconds
(C) 3 minutes and 27 seconds (D) 3 minutes and 39 seconds
(E) 3 minutes and 51 seconds

11. I have four consecutive odd numbers. The largest is one less than twice the smallest.

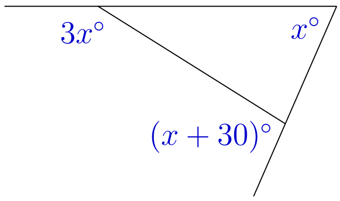
Which of the following is the largest of the four numbers?

(A) 9 (B) 11 (C) 13 (D) 15 (E) 21

12. On a compact disc, uncompressed music data is stored as 44 100 samples for each second of music, where each sample requires 4 bytes of data. Which of the following is closest to the number of bytes required to store 5 minutes of music on the disc?

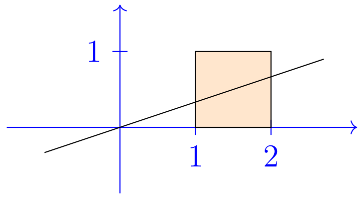
(A) 1 million (B) 5 million (C) 10 million (D) 50 million (E) 100 million

13. In the figure, the value of x is



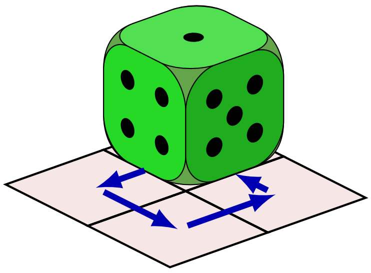
(A) 30 (B) 40 (C) 50 (D) 60 (E) 70

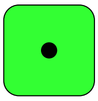
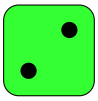
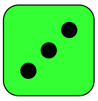
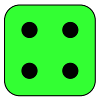
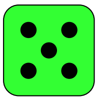
14. What is the equation of the line passing through $(0, 0)$ that bisects the square in the diagram?



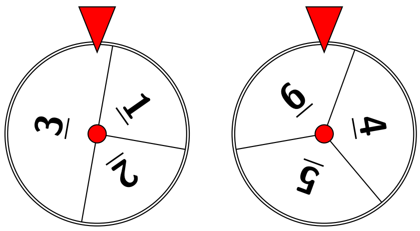
(A) $y = \frac{x}{3}$ (B) $y = \frac{x}{2}$ (C) $y = \frac{x}{4}$ (D) $y = 2x$ (E) $y = 3x$

15. A standard dice numbered 1 to 6 with opposite sides adding to 7 is placed on a 2 by 2 square as shown.
The dice is rolled over one edge onto each of the four base squares in turn and then back on to the original square, as indicated by the arrows.
Which side of the dice is now facing upwards?



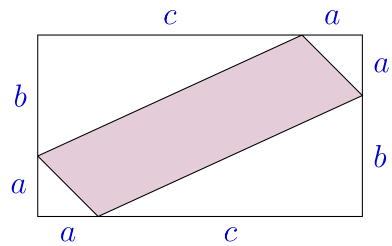
(A)  (B)  (C)  (D)  (E) 

16. The two spinners shown are spun and the numbers that the arrows point to when they stop are recorded.
For example, the numbers here are 3 and 6.
What is the probability that the sum of the two numbers is even?



(A) $\frac{1}{2}$ (B) $\frac{3}{8}$ (C) $\frac{3}{4}$ (D) $\frac{2}{3}$ (E) $\frac{5}{12}$

17. The area of the shaded region is given by



(A) $ab + ac$ (B) $a\sqrt{b^2 + c^2}$
(C) $bc + a^2 - ab - ac$ (D) $ab + ac - bc$
(E) $ab + ac - a^2$

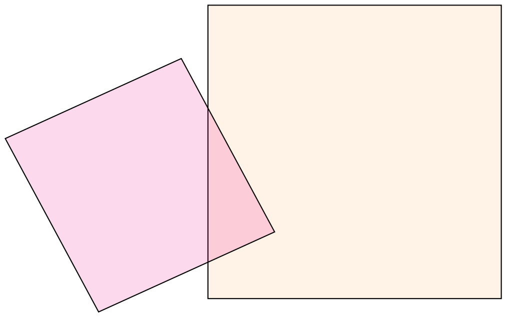
18. If k and n are positive integers, and $\sqrt{10n + k} = k$, then the smallest possible value for k is

(A) 3 (B) 4 (C) 5 (D) 6 (E) 10

19. Two squares are drawn as shown.

The smaller square covers $\frac{1}{8}$ of the larger square and the larger square covers $\frac{2}{9}$ of the smaller square.

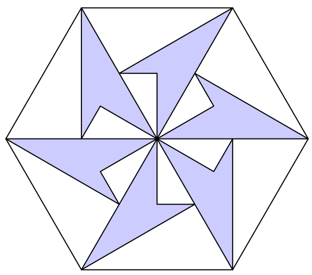
What is the ratio of the side length of the larger square to the side length of the smaller square?



(A) 3:2 (B) 7:3 (C) 7:4 (D) 5:3 (E) 4:3

20. Six identical darts fit inside a regular hexagon as shown. Each dart has three interior angles of $30°$, and one of $270°$.

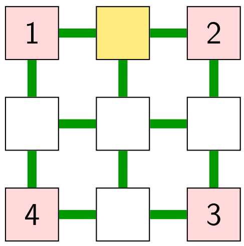
What fraction of the large hexagon is shaded?



(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{2}{5}$ (D) $\frac{4}{9}$ (E) $\frac{3}{8}$

21. We want to place numbers into each of the blank squares in this diagram so that each of the numbers we place is the average of those in the squares directly connected to it.

What number should we put in the middle square of the top row?



(A) $\frac{5}{3}$ (B) $\frac{3}{2}$ (C) $\frac{10}{9}$ (D) $\frac{11}{9}$ (E) $\frac{11}{6}$

22. To set the timer on his microwave oven, Rick enters the digits of the hours, minutes and seconds in order from left to right. For example, entering '12345' sets the timer to 1 hour 23 minutes 45 seconds, while entering '408' sets it to 4 minutes 8 seconds.

One day, Rick accidentally missed the last digit and the timer finished 4 minutes and 42 seconds earlier than he was expecting. What was the missing digit?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

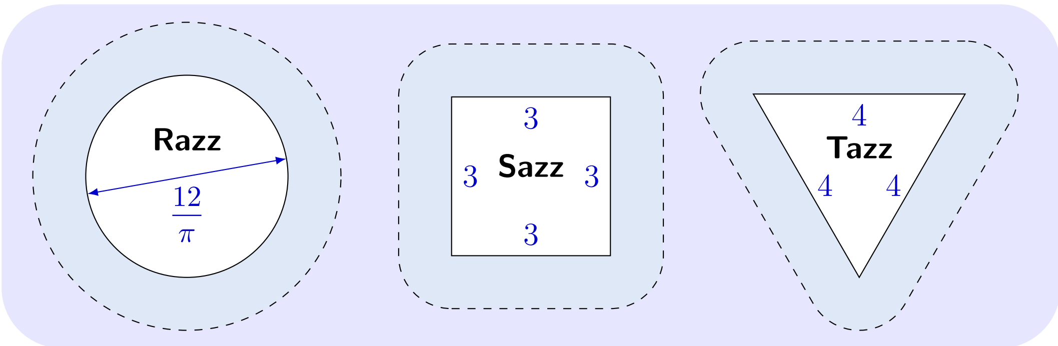
23. I build a large cube from unit cubes. Then I completely paint a number of faces of the large cube. When I dismantle the large cube, I find that I have 288 unit cubes without any paint on them. How many faces of the large cube were painted?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

24. The product $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{15^2}\right)$ is equal to

(A) $\frac{7}{13}$ (B) $\frac{8}{15}$ (C) $\frac{9}{16}$ (D) $\frac{10}{21}$ (E) $\frac{13}{24}$

25. Three artificial islands Razz, Sazz and Tazz were constructed in a shallow sea, each with a coastline of 12 km.



Around each island is a fishing zone, consisting of all points in the sea within 1 km of the island. Which islands have a fishing zone of the largest area?

(A) Razz only (B) Sazz only (C) Tazz only

(D) Sazz and Tazz (E) All three have the same area

26. In Australian Rules football, a team scores six points for a 'goal' and one point for a 'behind'. During a game, Vladislav likes to record his team's score with a sequence of sixes and ones. There are exactly three distinct sequences which give a final score of 7 points, namely 6,1 and 1,6 and 1,1,1,1,1,1,1.

How many different sequences give a final score of 20 points?

27. What is the smallest natural number n such that the number

$$N = 100000 \times 100002 \times 100006 \times 100008 + n$$

is a perfect square?

28. I have a large supply of matchsticks in four colours: red, yellow, blue and green. I use them to make squares where each side is one matchstick long.

I count two squares as the same if one can be rotated and/or reflected to match the shape and colour of the other.

How many different squares can be created?

29. Bluey divides the number 499 by each of the numbers $1, 2, 3, \ldots, 499$ and records the remainders in order. So her sequence begins:

$$0, 1, 1, 3, 4, 1, \ldots$$

Let M be the sum of these 499 remainders.

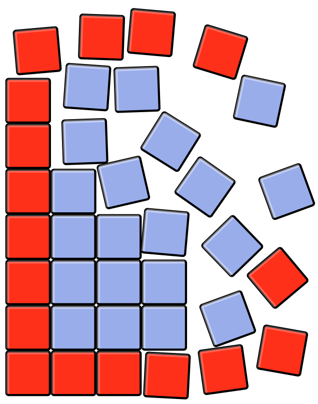
Jean-Luc divides the number 500 by each of the numbers $1, 2, 3, \ldots, 500$ and records the remainders in order. So his sequence begins:

$$0, 0, 2, 0, 0, 2, \ldots$$

Let N be the sum of these 500 remainders.

What is the difference between the numbers M and N?

30. Tyler has a large number of square tiles, all the same size. He has four times as many blue tiles as red tiles. He builds a large rectangle using all the tiles, with the red tiles forming a boundary 1 tile wide around the blue tiles.

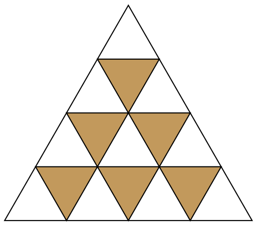


He then breaks up this rectangle and uses the tiles to make two smaller rectangles. Like the large rectangle, each of the smaller rectangles has four times as many blue tiles as red tiles, and the red tiles form a boundary 1 tile wide around the blue tiles.

How many blue tiles does Tyler have?

Senior Questions

1. Each small triangle is the same size.
What fraction of the largest triangle is shaded?



(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{2}{5}$ (E) $\frac{3}{8}$

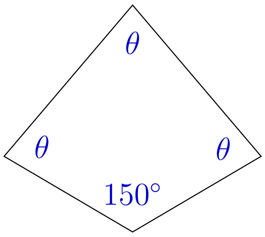
2. When 2021 is divided by 7 the remainder is

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

3. What is $12^1 - 12^{-1} - 12^0$?

(A) 0 (B) 1 (C) $10\frac{11}{12}$ (D) 11 (E) $11\frac{1}{12}$

4. In this diagram, what is the size of the angles marked θ?



(A) 70° (B) 75° (C) 80° (D) 85° (E) 90°

5. The value of $\dfrac{(20 \times 21) + 21}{21}$ is

(A) 20 (B) 21 (C) 22 (D) 41 (E) 42

6. Henry's electric scooter took him 1.5 km in 3 minutes and 45 seconds.
What was the average speed of Henry's trip in kilometres per hour?

(A) 20 (B) 21 (C) 24 (D) 25 (E) 30

7. $\dfrac{8^3 \times 3^6}{6^5} =$

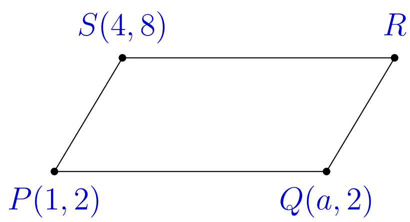
(A) 6 (B) 48 (C) 72 (D) 128 (E) 256

8. The product of recurring decimals $0.\dot{3}$ and $0.\dot{6}$ is the recurring decimal $0.\dot{x}$. What is the value of x?

(A) 1 (B) 2 (C) 5 (D) 7 (E) 9

9. The parallelogram shown has an area of 48 square units. The value of a is

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11



10. Mervin is allowed to paint the four walls and the ceiling of his rectangular bedroom as he wishes, subject to the following constraints. He paints each surface in one of three colours. He cannot paint two adjacent surfaces the same colour. He decides to use red, white and green. How many different ways can he paint his room?

(A) 2 (B) 3 (C) 6 (D) 12 (E) 24

11. Here is a list of fractions which, when written in simplest form, have a denominator less than 6:

$$\frac{1}{5}, \boxed{}, \frac{1}{3}, \frac{2}{5}, \boxed{}, \boxed{}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$$

The list is in ascending order, but three fractions are omitted. The sum of these three fractions is

(A) 1 (B) 2 (C) $\frac{21}{20}$ (D) $\frac{27}{20}$ (E) $\frac{29}{20}$

12. In autumn, Tilly's meadow changes rapidly with the weather.

The number of flowering plants starts at 150 000 but they are dying off so each week the number halves. At the same time, the number of fungi starts at 20 and triples each week.

To the nearest week, how long will it be until the fungi outnumber the flowering plants?

(A) 3 weeks (B) 5 weeks (C) 8 weeks (D) 11 weeks (E) 13 weeks

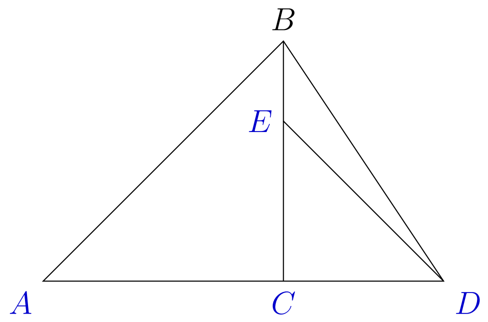
13. A formula in physics is given as:

$$r = \frac{mV}{qB}$$

If q was trebled, m was halved and r and B remained the same, then V would

(A) increase by a factor of 6 (B) decrease by a factor of 5 (C) stay the same (D) double (E) increase by a factor of 3

14. In this diagram, $AD = 12$ and ABC and CDE are right-angled isosceles triangles.
The area of triangle BDE is 9.
What is the area of triangle ABD?

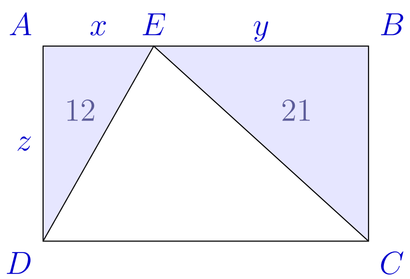


(A) 36 (B) 50 (C) 54 (D) 60 (E) 72

15. The numbers 1^{40}, 2^{30}, 3^{20} and 4^{10}, in increasing order, are

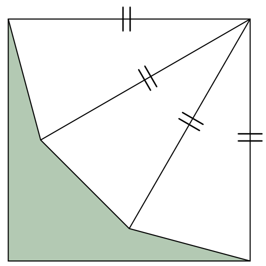
(A) $1^{40}, 4^{10}, 2^{30}, 3^{20}$ (B) $1^{40}, 3^{20}, 4^{10}, 2^{30}$ (C) $4^{10}, 3^{20}, 2^{30}, 1^{40}$ (D) $1^{40}, 2^{30}, 3^{20}, 4^{10}$ (E) $1^{40}, 2^{30}, 4^{10}, 3^{20}$

16. In the rectangle $ABCD$, the lengths marked x, y and z are positive integers.
Triangle AED has an area of 12 square units and triangle BCE has an area of 21 square units.
How many possible values are there for z?



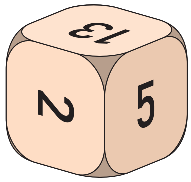
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

17. A square is divided into three congruent isosceles triangles and a shaded pentagon, as shown. What fraction of the square's area is shaded?



(A) $\frac{1}{3}$ (B) $\frac{1}{4}$ (C) $\frac{\sqrt{2}}{6}$ (D) $\frac{\sqrt{3}}{8}$ (E) $2 - \sqrt{3}$

18. A non-standard dice has the numbers 2, 3, 5, 8, 13 and 21 on it. The dice is rolled twice and the numbers are added together. What is the probability that the resulting sum is also a value on the dice?

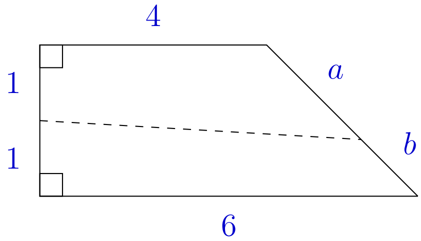


(A) $\frac{1}{9}$ (B) $\frac{5}{12}$ (C) $\frac{1}{3}$ (D) $\frac{1}{6}$ (E) $\frac{2}{9}$

19. If a and b are positive numbers, then $\sqrt{a^2+\frac{1}{b^2}} \times \sqrt{b^2+\frac{1}{a^2}}$ is equal to

(A) $\frac{a}{b}+\frac{b}{a}$ (B) $a^2b^2+\frac{1}{a^2b^2}$ (C) $ab+2+\frac{1}{ab}$ (D) $a+b+\frac{1}{a}+\frac{1}{b}$ (E) $ab+\frac{1}{ab}$

20. The quadrilateral shown is cut into two equal areas by the dashed line.
What is the ratio $a : b$?



(A) 2:1 (B) 7:3 (C) 5:2 (D) 4:3 (E) 3:2

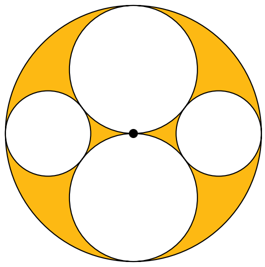
21. Positive integers x and y satisfy the equation

$$x^2+2xy+2y^2+2y=1988$$

What is the largest possible value of $x+y$?

(A) 33 (B) 38 (C) 42 (D) 46 (E) 47

22. What fraction of the area of the diagram is shaded?



(A) $\frac{1}{4}$ (B) $\frac{4}{9}$ (C) $\frac{5}{12}$ (D) $\frac{1}{2}$ (E) $\frac{5}{18}$

23. Sebastien is playing with a square paper serviette with side length 24 centimetres. He folds it in half along a diagonal to obtain a triangle ABC with a right angle at A. He then folds the triangle so that C ends up on line AB at some point D. Suppose that the fold created meets BC at the point X. Sebastien then folds B to meet X and notices that the fold created passes through the point D. The distance in centimetres between the points A and D is

(A) $6\sqrt{2}$ (B) $12(\sqrt{3}-1)$ (C) 10 (D) 12 (E) $24(\sqrt{2}-1)$

24. The fraction $\frac{a}{b}$ is positive and in lowest terms, so that a and b are positive with no common factors.

When I add the integer n to both the numerator and denominator of the fraction $\frac{a}{b}$, the result is double the original fraction.

When I subtract n from both the numerator and denominator, the result is triple the original value.

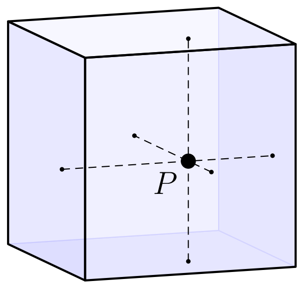
The value of n is

(A) 13 (B) 18 (C) 21 (D) 24 (E) 28

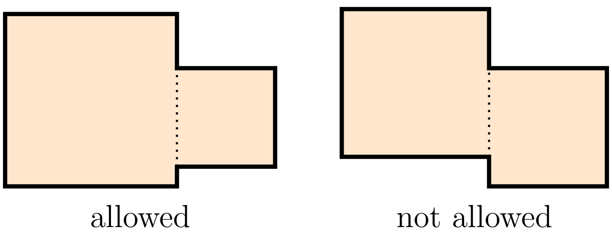
25. A cube has an internal point P such that the perpendicular distances from P to the six faces of the cube are 1 cm, 2 cm, 3 cm, 4 cm, 5 cm and 6 cm.

How many **other** internal points of the cube have this property?

(A) 5 (B) 11 (C) 23 (D) 47 (E) infinitely many



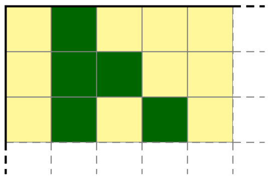
26. A 70 cm long loop of string is to be arranged into a shape consisting of two adjacent squares, as shown on the left. The side of the smaller square must lie entirely within the side of the larger one, so the example on the right is not allowed.



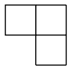
What is the minimum area of the resulting shape, in square centimetres?

27. How many pairs (m, n) exist, where m and n are different divisors of 2310 and n divides m? Both 1 and 2310 are considered divisors of 2310.

28. A grid that measures 20 squares tall and 21 squares wide has each of its squares painted either green or gold. The diagram shows part of the grid, including the top-left corner.



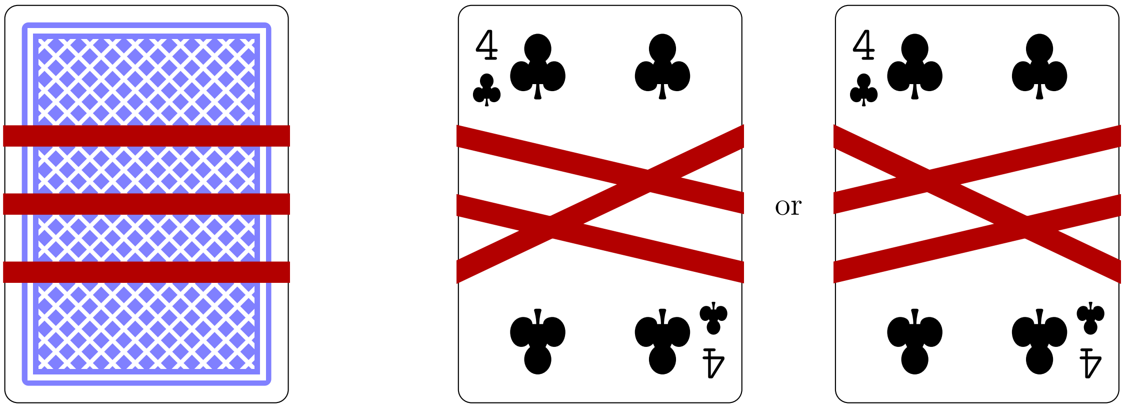
The pattern follows these rules:

- All squares in the leftmost column are gold.
- Only the second-leftmost square in the top row is green.
- For every triplet of squares in this orientation , the number of gold squares is odd.

How many of the $20 \times 21 = 420$ squares are painted green?

29. Starting with a paper rectangle measuring $1 \times \sqrt{2}$ metres, Sadako makes a single cut to remove the largest square possible, leaving a rectangle. She repeats this process with the remaining rectangle, producing another square and a smaller rectangle.
Since $\sqrt{2} \approx 1.41421356$ is irrational, she can in theory keep doing this forever, producing an infinite sequence of paper squares.
To the nearest centimetre, what would be the total perimeter of this infinite pile of squares?

30. An elastic band is wound around a deck of playing cards three times so that three horizontal stripes are formed on the top of the deck, as shown on the left. Ignoring the different ways the elastic band could overlap itself, there are essentially two different patterns it could make on the underside of the deck, as shown on the right.



Treating two patterns as the same if one is a $180°$ rotation of the other, how many different patterns are possible on the underside of the deck if the elastic band is wound around to form seven horizontal stripes on top?

Middle Primary Solutions

1. There are 7 dots,

hence (B).

2. $14 - 2 = 12$,

hence (C).

3. (Also UP1)

There are 3 equal parts and 2 are green, so $\frac{2}{3}$ is green,

hence (E).

4. $234 + 100 = 334$,

hence (D).

5. One hour is 60 minutes, half an hour is 30 minutes, and half of that is a quarter of an hour, which is 15 minutes,

hence (C).

6. There are 8 divisions on the scale, and a full tank holds 80 kL. So each division indicates 10 kL. Labelling these 10, 20, . . . , 80, the indicator is halfway between the 60 kL and 70 kL marker, giving a reading of 65 kL,

hence (D).

7. (Also UP2)

The unknown number is 5 more than 9, which is 14,

hence (E).

8. *Alternative 1*

A cube has 6 faces, so there are $6 \times 4 = 24$ small squares in total,

hence (D).

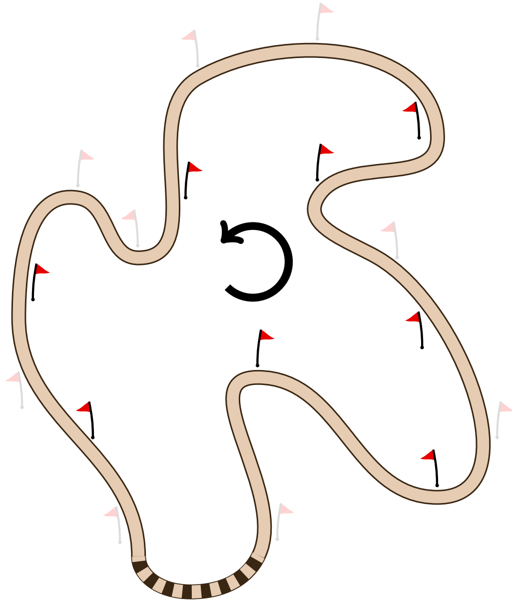
Alternative 2

We can see 12 squares on the three faces shown in the diagram. There are another 12 squares on the three hidden faces, so there are $12 + 12 = 24$ squares in total,

hence (D).

9. Think of the track as a closed loop, with participants running around it in an anti-clockwise direction.

Flags that are on the left side of the track when a participant runs past must be inside the closed loop, and there are 8 such flags as shown,



hence (B).

10. We can measure each shaded area using the equilateral triangles in the grid as units:

$$A = 16\ , \quad B = 14 + \frac{2}{2} = 15\ , \quad C = 10\ , \quad D = 12 + \frac{4}{2} = 14\ , \quad E = 12 + \frac{2}{2} = 13$$

Then $A = 16$ is the largest,

hence (A).

11. (Also UP8)
There are 4 students ahead of him, 8 behind him, and Leo himself. So there are $4+1+8 = 13$ students in line,

hence (E).

12. (Also UP11, J10)
In (B), the five cards can't be dropped in any order, since 2, 7, 6, 4, 9 is a cycle of 5 cards where each is on top of the next one, so no single card is on top.

Several of the other options have cycles with fewer cards: (A) has 2, 9, 4; (C) has 4, 9, 7, 6 and (D) has 2, 7, 6.

The remaining option is (E), which is possible in the order 4, 7, 9, 2, 6 or in the order 7, 4, 9, 2, 6,

hence (E).

Note: Another way of eliminating (B) and (D) is to observe that the last card dropped must be on top, not under another card. However, in (B) each card is under another card, and likewise for (D). Further, (A) and (C) can also be eliminated. Each has a top (last) card, but once this is removed none of the other four cards is on top, so none of these can be the second-last card.

13. There are 24 quarters and four people, so each gets six quarters. Then $\frac{6}{4} = \frac{3}{2} = 1\,\frac{1}{2}$,

hence (E).

14. From the four diagrams the boxes that area heavier than another box are A, D, C and E. So B is the lightest box,

hence (B).

15. Lydia needs \$28, since then her grandfather will pay the other \$28.
So she has to save another $28 - 16 = 12$ dollars,

hence (B).

16. The largest numbers that can be made with the 5 digits are, in descending order,

$$94321, 94312, 94231, 94213, 94132, 94123, \ldots$$

The first even number in the list is 94312, which has 1 in the tens place,

hence (A).

17. In column 4, the total of 5 can only be made as $2+1+1+1$, so $F = 2$ and $H = 1$.
In row 4, the total is 13 and $H = 1$. Then three R's add to 12, and so $R = 4$.
In row 1, the total is 16 and $F + F = 2 + 2 = 4$. Then two M's add to 12, and so $M = 6$.
In column 1, the total is 16 and $M + R = 6 + 4 = 10$. Then two L's add to 6, and so $L = 3$,

hence (C).

18. Currently their ages add to $19 + 26 + 31 = 76$, which is 24 less than 100. Each year, the sum of their ages goes up by 3, so it will take $24 \div 3 = 8$ years for their ages to add up to 100,

hence (B).

19. From the ends of the labels we can tell that Ainslie is the 3rd column, Turner and Downer are the 1st and 4th columns, or vice versa, and Watson and Dickson are the 2nd and 5th columns, or vice versa.

Downer must have a taller column than Watson, so the only possibility is that they are the 1st and 5th columns, respectively. This means Turner is the 4th column and therefore represents 3 students,

hence (A).

20. (Also UP19)
To get a units digit of 2 in the answer, the units digit of the first term must be 4. There is a carry of 1 into the tens column, so to get a carry of 1 from the tens into the hundreds, the tens digit of the second term must be at least 6. This results in the following four possibilities:

$$\begin{array}{r} 34 \\ +\,68 \\ \hline 102 \end{array} \qquad \begin{array}{r} 34 \\ +\,78 \\ \hline 112 \end{array} \qquad \begin{array}{r} 34 \\ +\,88 \\ \hline 122 \end{array} \qquad \begin{array}{r} 34 \\ +\,98 \\ \hline 132 \end{array}$$

hence (C).

21. Looking at the last word, it looks like 'apples' = 'kete'. Since there is no other word that appears three times, this is the only possibility. Then 'oranges' = 'kito' and the second English sentence matches the first Windarian sentence.

From that sentence it looks like 'likes' = 'bem' and 'Dad' = 'Ato', which must be true, since 'likes' appears in two sentences, whereas 'Dad' doesn't.

Consequently, the first English sentence matches the third Windarian sentence, which means that 'Mum' = 'Eke'. Finally, 'loves' = 'tum', so 'Mum loves oranges' = 'Eke tum kito',

hence (C).

22. *Alternative 1*

We want 6 consecutive numbers that add to 147. Since $6 \times 20 = 120$ and $6 \times 30 = 180$, we guess that the numbers are in the 20s.

As a first try, $20+21+22+23+24+25 = 135$. From here $21+22+23+24+25+26 = 141$ (replace 20 by 26) and $22+23+24+25+26+27 = 147$ (replace 21 by 27),

hence (B).

Alternative 2

The average of the six page numbers is $147 \div 6 = 24.5$. The difference between the first and last pages is 5 pages. So we try 2.5 either side of 24.5: the first page number is 22 and the last is 27. Checking, $22+23+24+25+26+27 = 147$,

hence (B).

23. When the cubes are joined, there are 5 pairs of faces joined to hold the 6 cubes in one piece. So there are 10 faces that are not exposed to the outside, and these 10 faces do not get any blue paint,

hence (D).

24. (Also UP23)

Half a rotation of the gear M means that the gear rotates clockwise by 5 teeth.

Gears A and C also rotate by 5 teeth, but since they have 20 teeth, this is a quarter-turn. Also A and C will move in the opposite direction to gear M, so they will both rotate 90 degrees anticlockwise.

Only diagram (A) shows this,

hence (A).

25. There are 6 choices for the triple. For each choice of the triple, there are 5 choices for the pair. So there are $6 \times 5 = 30$ possible full house rolls,

hence (E).

26. From the complete diagonal, the magic number is $16 + 10 + 7 + 1 = 34$.
The bottom row has $4 + 15 + 1 = 20$, so needs $34 - 20 = 14$.
The last column has $8 + 12 + 1 = 21$, so needs $34 - 21 = 13$.
Then the top row has $16 + 2 + 13 = 31$, so needs $a = 34 - 31 = 3$.
The third column has $2 + 7 + 14 = 23$, so needs $c = 34 - 23 = 11$.
The second column has $3 + 10 + 15 = 28$, so needs $b = 34 - 28 = 6$.
The remaining two squares are not needed but can be filled in.
The product is $a \times b \times c = 3 \times 6 \times 11 = 198$,

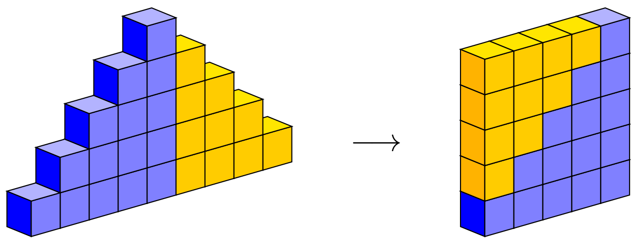
16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

hence (198).

27. Hayden has $1420 - 505 = 915$ dollars more than Mitchell, both before and after spending. After spending, Hayden has 4 times as much as Mitchell, so the difference is 3 times as much as Mitchell. So Mitchell's remaining amount is $915 \div 3 = 305$ dollars. Since he started with \$505, he must have spent \$200,

hence (200).

28. *Alternative 1*
The numbers of blocks in each tower seem to be consecutive square numbers: 1, 4, 9. This can be confirmed by rearranging each tower into a square. For instance, here is the 5th tower arranged into a 5×5 square.



Then the number of blocks in the first ten towers is the sum of the first 10 square numbers

$$1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 = 385$$

hence (385).

Alternative 2
Considering all ten towers together, we can count the number of rows of each length.
All ten towers have a row of length 1, using $10 \times 1 = 10$ blocks.
Nine of the towers have a row of length 3, using $9 \times 3 = 27$ blocks.
Eight of the towers have a row of length 5, using $8 \times 5 = 40$ blocks.
Following this pattern, the total number of blocks used is

$$\begin{aligned} &10\times1 + 9\times3 + 8\times5 + 7\times7 + 6\times9 + 5\times11 + 4\times13 + 3\times15 + 2\times17 + 1\times19 \\ &= 10 + 27 + 40 + 49 + 54 + 55 + 52 + 45 + 34 + 19 = 385 \end{aligned}$$

hence (385).

29. The second number must end in 5.

The third number uses digits from $\{1, 2, 3, 4, 6\}$ and is a multiple of 6 but not a multiple of 4. Two-digit multiples of 6 are 12, 18, 24, 30, 36, 42, 48, 54, 60 and 66. Of these, only 42 uses the digits available and is not a multiple of 4.

The first number uses digits from $\{1, 3, 6\}$ and is a multiple of 4 but not a multiple of 6. This must be 16.

Returning to the second number, it must be 35.

So her three numbers are 16, 35 and 42, and then $16 \times 35 + 42 = 560 + 42 = 602$,

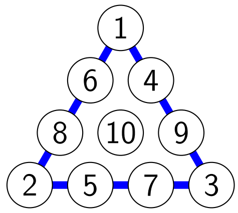
hence (602).

30. (Also UP29)

The total of all three sides includes the three corner numbers twice each, the six edge numbers once each and the centre number zero times. So to make this *triple total* as small as possible, we look for a solution where the largest number (10) is in the centre, the smallest numbers are at the corners $(1 + 2 + 3 = 6)$ and the remaining six numbers are on the edges $(4 + 5 + 6 + 7 + 8 + 9 = 39)$.

Then this triple total would be $2 \times 6 + 39 = 51$ and the side total would be $51 \div 3 = 17$.

With this information, many solutions are possible, just by placing the numbers to obtain a total of 17 on each side. One is shown here, confirming that a side total of 17 is the smallest possible,



hence (17).

Upper Primary Solutions

1. (Also MP3)

There are 3 equal parts and 2 are green, so $\frac{2}{3}$ is green,

hence (E).

2. (Also MP7)

The unknown number is 5 more than 9, which is 14,

hence (E).

3. The perimeter is $4 + 4 + 4 + 9 = 21$ centimetres,

hence (E).

4. Writing as fractions, the numbers in (A)–(E) are $\frac{2}{10000}, \frac{2}{1000}, \frac{2}{100}, \frac{2}{10}, \frac{2}{1}$, and the smallest is the one with the largest denominator. This is $\frac{2}{10000}$,

hence (A).

5. $\frac{1}{2} + \frac{2}{4} - \frac{4}{8} = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \frac{1}{2}$,

hence (A).

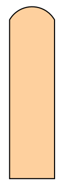
6. Suri cannot make 30 cents. She cannot use a 50c coin since this is already too much. So she must only use 20c coins, but then she can only make multiples of 20 cents.

All of the other amounts are possible: 50 cents from one 50c coin, 60 cents from three 20c coins, 80 cents from four 20c coins and 70 cents from one of each coin,

hence (D).

7. After cutting, the sheet of paper will be made from a number of panels each of which is a long rectangle with a curved top as shown.

Only (A) is made of these panels,



hence (A).

8. (Also MP11)

There are 4 students ahead of him, 8 behind him, and Leo himself. So there are $4+1+8 = 13$ students in line,

hence (E).

9. Thyme and fennel are equal, so thyme also weighs 5 grams. Sage weighs as much as fennel and thyme combined, which is 10 grams. Mint weighs as much as fennel, thyme and sage combined, which is 20 grams,

hence (D).

10. For each pear there are 2 nectarines and for these 2 nectarines there are 4 apples, making 7 pieces of fruit.

Since there are 14 pieces of fruit, there must be twice this, so there are 2 pears, 4 nectarines and 8 apples,

hence (D).

11. (Also MP12, J10)

In (B), the five cards can't be dropped in any order, since 2, 7, 6, 4, 9 is a cycle of 5 cards where each is on top of the next one, so no single card is on top.

Several of the other options have cycles with fewer cards: (A) has 2, 9, 4; (C) has 4, 9, 7, 6 and (D) has 2, 7, 6.

The remaining option is (E), which is possible in the order 4, 7, 9, 2, 6 or in the order 7, 4, 9, 2, 6,

hence (E).

Note: Another way of eliminating (B) and (D) is to observe that the last card dropped must be on top, not under another card. However, in (B) each card is under another card, and likewise for (D). Further, (A) and (C) can also be eliminated. Each has a top (last) card, but once this is removed none of the other four cards is on top, so none of these can be the second-last card.

12. The two squares have an area of 25 square centimetres, so their sides are 5 cm long. The rectangle has sides of 5 and 10 centimetres, and so its perimeter is $5 + 10 + 5 + 10 = 30$ cm,

hence (D).

13. For each hop, the distance between the kangaroo and the wallaby reduces by 3 metres. So the number of hops to catch up is $42 \div 3 = 14$,

hence (D).

14. The total amount of wire is 50 centimetres and $8 + 8 + 3 + 3 = 22$ centimetres of this is vertical. So 28 centimetres of wire is horizontal, which is the value x in the diagram,

hence (A).

15. Margie thinks her watch is 10 minutes fast, so she will turn up at the station when the time on her watch is 10:10 am, as she thinks that would actually be at 10 am.

Since her watch is 10 minutes slow, this actually at 10:20 am, 20 minutes after the train left. So she misses the train by 20 minutes.

Rosie thinks her watch is 10 minutes slow, so she will turn up at the station when the time on her watch is 9:50 am. Since her watch is fast, this is at 9:45 am. That is, she arrives early and has to wait 15 minutes before catching the train.

So only Margie misses the train, by 20 minutes,

hence (B).

16. *Alternative 1*

The number of blocks in the squares go up in fours, since the number of blocks on each side goes up in ones.

The pattern in the number of blocks is then 8, 12, 16, 20, 24, 28, 32, with the 7th value being 32,

hence (C).

Alternative 2

Hollow square 7 has a 7×7 hole in the middle of a 9×9 square. The number of blocks in hollow square 7 is $81 - 49 = 32$,

hence (C).

17. There is 50 mL of olive oil. For this to be $\frac{1}{3}$ of the mixture there must be 100 mL of vinegar, so I must add another 50 mL of vinegar,

hence (C).

18. Calculating $2021 \div 24$ gives 84 with remainder 5, so 2021 hours is 84 days and 5 hours. Starting from 10 am we just add the 5 hours to the time of 10 am, giving 3 pm,

hence (C).

19. (Also MP20)

To get a units digit of 2 in the answer, the units digit of the first term must be 4. There is a carry of 1 into the tens column, so to get a carry of 1 from the tens into the hundreds, the tens digit of the second term must be at least 6. This results in the following four possibilities:

$$\begin{array}{r} 34 \\ +68 \\ \hline 102 \end{array} \qquad \begin{array}{r} 34 \\ +78 \\ \hline 112 \end{array} \qquad \begin{array}{r} 34 \\ +88 \\ \hline 122 \end{array} \qquad \begin{array}{r} 34 \\ +98 \\ \hline 132 \end{array}$$

hence (C).

20. The quiz night will have smallest number of teams when the fewest people attend, arranged into large teams. $25 \div 8 = 3\frac{1}{8}$ so 3 teams is too few, and 4 is possible.

The largest number of teams is when the most people attend, arranged into small teams. $35 \div 6 = 5\frac{5}{6}$ so 6 is too many teams but 5 is possible,

hence (A).

21. *Alternative 1*

Let the candidates be Annie, Bonnie, Connie and Donny. Give Annie 31 votes and Donny 0 votes. Then give Bonnie $31 - 19 = 12$ votes and Connie $31 - 25 = 6$ votes. These four amounts have the right margins, but only use $31 + 12 + 6 = 49$ votes.

With the remaining $453 - 49 = 404$ votes, give each candidate $404 \div 4 = 101$ votes. This keeps the margins the same and uses all 453 votes. Then Annie, the winner, receives $31 + 101 = 132$ votes,

hence (E).

Alternative 2

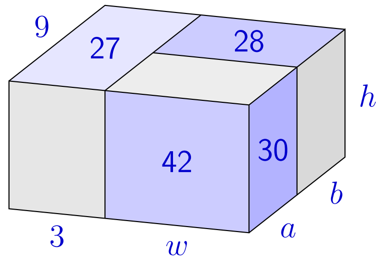
If there were an additional $31 + 25 + 19 = 75$ voters, then the 3 losing candidates could have a number of votes equal to the winner.

This would require $453 + 75 = 528$ voters, and give $528 \div 4 = 132$ votes each.

Removing these imagined 75 voters, the winner still has 132 votes,

hence (E).

22. (Also J16)



The top face of the left block is 3 cm wide with area $27\,\text{cm}^2$, so is 9 cm deep, as labelled on the diagram.

On this diagram, the solution $h = 6$, $w = 7$, $a = 5$ and $b = 4$ can be found by exploring possible factorisations. However, there may be other solutions.

The height h is a factor of both 30 and 42, which means $h = 1$, 2, 3 or 6. Then the depth of the front-right block is $a = 30$, 15, 10 or 5. However, a must be less than 9, so the only possibility is $a = 5$. Then $h = 6$, $b = 4$ and $w = 7$, and so our first solution is the only solution.

The combined volume is $(3 + 7) \times 9 \times 6 = 10 \times 9 \times 6 = 540\,\text{cm}^3$,

hence (B).

23. (Also MP24)
Half a rotation of the gear M means that the gear rotates clockwise by 5 teeth.

Gears A and C also rotate by 5 teeth, but since they have 20 teeth, this is a quarter-turn. Also A and C will move in the opposite direction to gear M, so they will both rotate 90 degrees anticlockwise.

Only diagram (A) shows this,

hence (A).

24. The completely green tiles form an inner rectangle consisting of all tiles that are not edge or corner tiles, inside the outer rectangle.

This inner rectangle has 20 tiles, so it can be 1×20, 2×10 or 4×5.

Then the outer rectangle can be 3×22, 4×12 or 6×7, with 66, 48 or 42 tiles. Of these, the most tiles that can be used is 66,

hence (B).

25. The numbers of students in each column are 7, 9, 6, 3 and 5, respectively, so there are 30 students in Jeremy's class.

Then North and Central will include half the class, which is 15 students. The only two amounts that add to 15 are $9 + 6$ in the second and third columns, so North and Central are 9 and 6 or vice versa.

Since South is more common than Central, Central can't be 9. So North is column 2 (9) and Central is column 3 (6). Then South must be column 1 (7).

This leaves East and West as 3 and 5, or vice versa, but both of these are less than 7 so either way East will be less than South, which is consistent with the only other piece of information. Thus Jeremy is only able to determine the columns for North, South and Central Windar, but not East and West,

hence (D).

26. Initially there is 1 square, so she needs 999 more. With each pair of cuts, the number of squares increases by 3. So to get to 1000 squares, Pip needs to perform a pair of cuts 333 times. Therefore she needs to make 666 cuts in total,

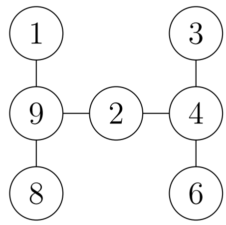
hence (666).

27. Let the product of the numbers in each line be X.

Thinking through the numbers from 1 to 9, there is only one multiple of 7, which is 7 itself. If 7 were a factor of X, there would need to be at least two multiples of 7 in the diagram, one in the left column and one in the right column. This is impossible, so 7 is one of the numbers that is not used. By the same reasoning, 5 is the other number that is not used.

So the seven numbers that are used are $1, 2, 3, 4, 6, 8, 9$. Consequently X is a multiple of each of these numbers, which means that X is a multiple of their least common multiple, which is 72.

Try $X = 72$. Possible products making 72 with three of the seven numbers are $72 = 1 \times 8 \times 9 = 2 \times 4 \times 9 = 3 \times 4 \times 6$. These can be arranged in the diagram as shown. So $X = 72$ is possible.



To see that no larger multiple of 72 is possible, consider $X = 144$. If this were possible, the two vertical lines would multiply to 144^2. However, all seven numbers only multiply to

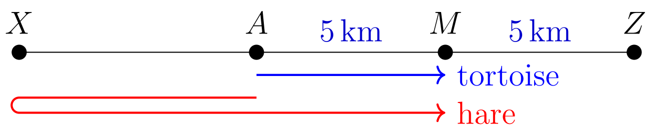
$$(2 \times 8 \times 9) \times (3 \times 4 \times 6) = 144 \times 72$$

which is too small. Similarly any larger multiple of 72 cannot be reached.

Consequently the only possible value of the product is $X = 72$,

hence (72).

28. Let the race be from A to Z, with midpoint M and let X be the point where the hare turns around:



The tortoise travels 5km and the hare 10 times as far, so 50km. So the hare's run from A to X to A is 45 km. Since the hare runs at 30 km/h, this takes 1.5 hours, or 90 minutes.

The time the hare runs in the wrong direction is half this, or 45 minutes,

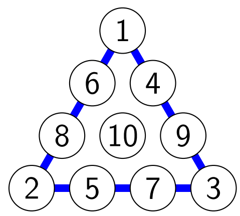
hence (45).

29. (Also MP30)

The total of all three sides includes the three corner numbers twice each, the six edge numbers once each and the centre number zero times. So to make this *triple total* as small as possible, we look for a solution where the largest number (10) is in the centre, the smallest numbers are at the corners $(1 + 2 + 3 = 6)$ and the remaining six numbers are on the edges $(4 + 5 + 6 + 7 + 8 + 9 = 39)$.

Then this triple total would be $2 \times 6 + 39 = 51$ and the side total would be $51 \div 3 = 17$.

With this information, many solutions are possible, just by placing the numbers to obtain a total of 17 on each side. One is shown here, confirming that a side total of 17 is the smallest possible,



hence (17).

30. *Alternative 1*

Let the numbers be A and B where Oliver made the mistake in A.

He accidentally made A one-tenth as big as it should have been. Another way of seeing this is that he reduced its value by $\frac{9}{10}$ of A, so he reduced the sum $A + B$ by $\frac{9}{10}$ of A.

The reduction in the sum $A + B$ from what it should have been to what Oliver calculated was $11.63 - 5.87 = 5.76$. This is $\frac{9}{10}$ of A. Then $\frac{1}{10}$ of A is $5.76 \div 9 = 0.64$ and so $A = 6.4$.

Finally, $B = 11.63 - 6.4 = 5.23$ and $A - B = 6.4 - 5.23 = 1.17$. Multiplying this by 100 gives 117,

hence (117).

Alternative 2

Lay out the addition as it should have been and as Oliver did it:

$$
\begin{array}{cccccc}
 & & \square & . & \square & \square \\
+ & \square & \square & . & \square & \\
\hline
 & 1 & 1 & . & 6 & 3
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & \square & . & \square & \square \\
+ & \square & . & \square & \square \\
\hline
 & 5 & . & 8 & 7
\end{array}
$$

Then make some deductions based on adding each column, while keeping the digits in both versions of the calculation the same:

$$
\begin{array}{cccccc}
 & & \square & . & \square & 3 \\
+ & \square & \square & . & 4 & \\
\hline
 & 1 & 1 & . & 6 & 3
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & \square & . & \square & 3 \\
+ & \square & . & \square & 4 \\
\hline
 & 5 & . & 8 & 7
\end{array}
$$

$$
\begin{array}{cccccc}
 & & \square & . & 2 & 3 \\
+ & \square & 6 & . & 4 & \\
\hline
 & 1 & 1 & . & 6 & 3
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & \square & . & 2 & 3 \\
+ & \square & . & 6 & 4 \\
\hline
 & 5 & . & 8 & 7
\end{array}
$$

$$
\begin{array}{cccccc}
 & & 5 & . & 2 & 3 \\
+ & 0 & 6 & . & 4 & \\
\hline
 & 1 & 1 & . & 6 & 3
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & 5 & . & 2 & 3 \\
+ & 0 & . & 6 & 4 \\
\hline
 & 5 & . & 8 & 7
\end{array}
$$

Then $6.4 - 5.23 = 1.17$ and $100 \times 1.17 = 117$,

hence (117).

Junior Solutions

1. $2021 - 1202 = 819$,

hence (C).

2. The horizontal segments are $6 + 2 + 1 + 1 + 4 = 14$.
The vertical segments are $2 + 1 + 2 + 2 + 5 = 12$.
Therefore the total perimeter is $14 + 12 = 26$,

hence (B).

3. With $b = 6$ and $h = 4$, the area is $A = \frac{1}{2} \times b \times h = \frac{1}{2} \times 6 \times 4 = 12\,\text{cm}^2$,

hence (B).

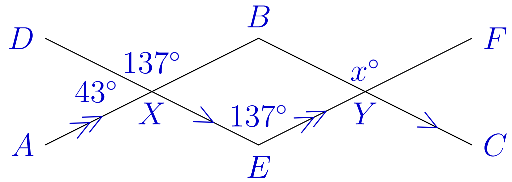
4. There are 4 intervals between 0 and 1, so these are quarters. Each quarter can be each cut in half to make eighths, and then $\frac{3}{8}$ is between $\frac{1}{4} = Q$ and $\frac{1}{2} = R$,

hence (B).

5. The numbers are (A) 20 200, (B) 22 000, (C) 2020, (D) 202 and (E) 220. Of these, the closest to 2021 is 2020,

hence (C).

6. Let X and Y be the points of intersection, as shown.
Due to supplementary angles, $\angle DXB = 180° - 43° = 137°$.
Due to corresponding angles, $\angle DEF = \angle DXB = 137°$.
Due to corresponding angles $x° = \angle BYF = \angle DEF = 137°$.



hence (E).

7. The correct calculation is $5 \times 2 + 4 = 10 + 4 = 14$, but Mr Meow's calculation is $5 + 2 \times 4 = 5 + 8 = 13$, which is one less,

hence (B).

8. From 11:49 am, there are 11 minutes until midday. Since $75 - 11 = 64$, the cake should come out of the oven at 1:04 pm,

hence (A).

9. The number of people who now know the joke is $1 + 3 + 3 \times 3 = 13$,

hence (C).

10. (Also MP12, UP11)

In (B), the five cards can't be dropped in any order, since 2, 7, 6, 4, 9 is a cycle of 5 cards where each is on top of the next one, so no single card is on top.

Several of the other options have cycles with fewer cards: (A) has 2, 9, 4; (C) has 4, 9, 7, 6 and (D) has 2, 7, 6.

The remaining option is (E), which is possible in the order 4, 7, 9, 2, 6 or in the order 7, 4, 9, 2, 6,

hence (E).

Note: Another way of eliminating (B) and (D) is to observe that the last card dropped must be on top, not under another card. However, in (B) each card is under another card, and likewise for (D). Further, (A) and (C) can also be eliminated. Each has a top (last) card, but once this is removed none of the other four cards is on top, so none of these can be the second-last card.

11. (Also I8)

Alternative 1

There are four bags, and one-fifth of a bag is lucerne. Let the lucerne be one unit, then one bag has 5 units and four bags has 20 units. So the lucerne is 1 part in 20 in the total mix, which is 5%,

hence (B).

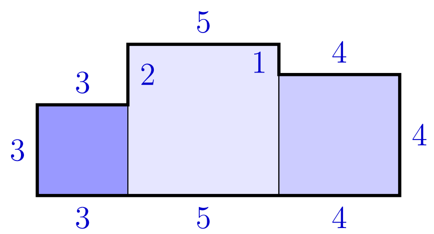
Alternative 2

Each bag is one-quarter or 25% of the mix. The lucerne is one-fifth of one bag, which is 5% of the mix,

hence (B).

12. *Alternative 1*

The squares have sides 3 cm, 5 cm and 4 cm, so we can find all lengths in the perimeter:



Then the perimeter is $3 + 3 + 3 + 2 + 5 + 1 + 4 + 4 + 4 + 5 = 34$,

hence (A).

Alternative 2

The original squares have combined perimeter $12+20+16 = 58$ cm and sides 3 cm, 5 cm and 4 cm. The join on the left is 3 cm long, so reduces the combined perimeter by $2 \times 3 = 6$ cm. Similarly the join on the right reduces the combined perimeter by 8 cm. The perimeter of the shape is then $58 - 6 - 8 = 34$ cm,

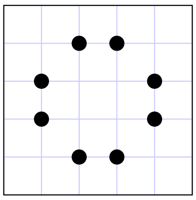
hence (A).

13. The next palindrome after 199 786 is 199 991. Then $199\,991 - 199\,786 = 991 - 786 = 205$,

hence (D).

14. The sum of all four distances, which is 10 cm, is equal to the width plus the height of the square, which is twice length of the square's side. So the square has sides of length 5 cm.

The diagram suggests that there are 8 possible points. Indeed, if we choose one side to be of distance 1 (a choice from 4 sides) and an adjacent side to be of distance 2 (a choice from 2 sides), there is a unique position for the point. There are $4 \times 2 = 8$ such choices.



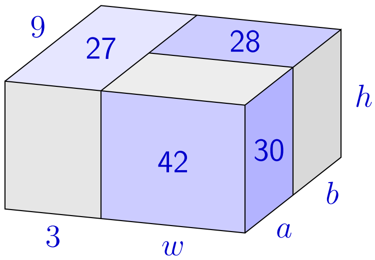
From these 8 possible points, one of them is P, so there are 7 other points with these distances,

hence (D).

15. Consider $x = \dfrac{\triangle}{10}$. If x is less than $\dfrac{2}{3}$, then $x + \dfrac{1}{3}$ will be less than 1; and if x is greater than or equal to $\dfrac{2}{3}$, then $x + \dfrac{1}{3}$ will be greater than or equal to 1. So we just consider whether $\dfrac{\triangle}{10} < \dfrac{2}{3}$. But $\dfrac{2}{3} = 0.\dot{6} \approx 0.67$ so that $\triangle = 1, 2, 3, 4, 5, 6$ are the possible values,

hence (D).

16. (Also UP22)



The top face of the left block is 3 cm wide with area 27 cm^2, so is 9 cm deep, as labelled on the diagram.

On this diagram, the solution $h = 6$, $w = 7$, $a = 5$ and $b = 4$ can be found by exploring possible factorisations. However, there may be other solutions.

The height h is a factor of both 30 and 42, which means $h = 1$, 2, 3 or 6. Then the depth of the front-right block is $a = 30$, 15, 10 or 5. However, a must be less than 9, so the only possibility is $a = 5$. Then $h = 6$, $b = 4$ and $w = 7$, and so our first solution is the only solution.

The combined volume is $(3 + 7) \times 9 \times 6 = 10 \times 9 \times 6 = 540\,\text{cm}^3$,

hence (B).

17. (Also I11)

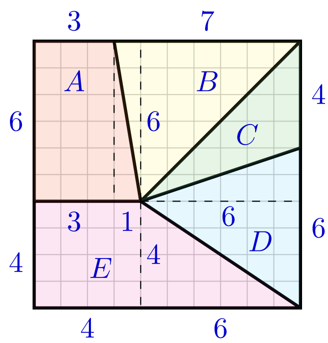
Let x be the smallest number, so that the four numbers are $x, x + 2, x + 4$ and $x + 6$.

Then $x + 6 = 2x - 1$. Solving, $x + 7 = 2x$ and then $x = 7$.

Consequently the numbers are $7, 9, 11, 13$ and the largest is 13,

hence (C).

18. Label some more side lengths, then evaluate the areas, using the grid shown and area formulas for rectangles and triangles:



$$A = 3 \times 6 \ + \ \frac{1}{2} \times 1 \times 6 = 18 + 3 = 21\,\text{m}^2$$

$$B = \frac{1}{2} \times 7 \times 6 = 21\,\text{m}^2$$

$$C = \frac{1}{2} \times 4 \times 6 = 12\,\text{m}^2$$

$$D = \frac{1}{2} \times 6 \times 6 = 18\,\text{m}^2$$

$$E = 4 \times 4 \ + \ \frac{1}{2} \times 6 \times 4 = 16 + 12 = 28\,\text{m}^2$$

Then region E has the largest area,

hence (E).

19. *Alternative 1*

Together, Rachel and Sandy have 180 cars. Thandie has the other $300 - 180 = 120$.
Together, Rachel and Thandie have 200 cars. So Rachel has $200 - 120 = 80$ cars,

hence (A).

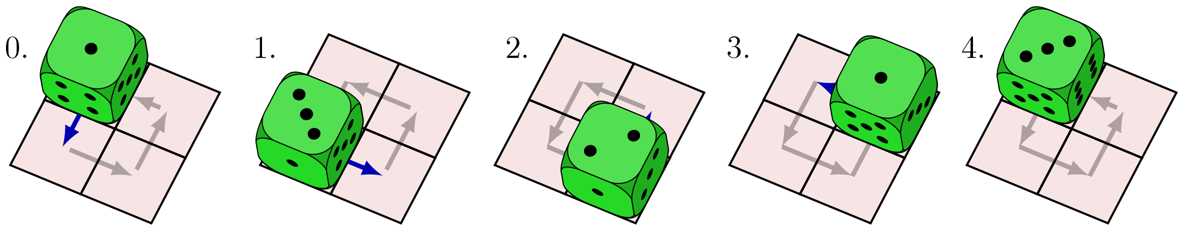
Alternative 2

Let the number of cars that Rachel, Sandy and Thandie have be R, S and T.
Then $R + S + T = 300$, $R + S = 180$ and $R + T = 200$.
Then $R+S+R+T = 180+200$ so that $2R+S+T = 380$. Subtracting $R+S+T = 300$ gives $R = 80$,

hence (A).

20. (Also I15)

Here is the result of rolling the dice. Note that the corner of the dice in the centre of the 2×2 square remains there, and it is the corner common to the faces with 4, 5, and 6 pips.



Then the upward-facing side is ,

hence (C).

21. A puzzle with two clues is possible: place 1 and 3 in opposite corners as shown, then the top row can't be 1,2,3 so must be 1,3,2. Similarly the bottom row must be 2,1,3. Finally the middle row must be 3,2,1.

clues →

1		
		3

1	3	2
2	1	3

1	3	2
3	2	1
2	1	3

A puzzle with only one clue will not have a unique solution. This is because a solution for this clue can always be turned into a different solution for the same clue. For instance, if we rewrite our solution but completely swap the two rows that don't include the clue, we get a different solution.

clue →

2		

1	2	3
2	3	1
3	1	2

3	1	2
2	3	1
1	2	3

So the smallest number of clues that Leonhard could include is 2,

hence (B).

22. The seating has to be cAcAc where A means adult and c means child.

The adults can be placed two ways.

For each of these two ways, the grandchildren can be placed 6 ways: 3 choices for the first child, then 2 choices for the second and third children.

In all there are 12 arrangements,

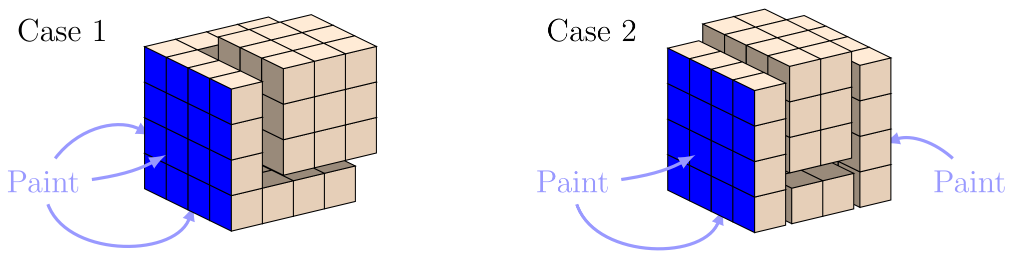
hence (B).

23. There are two cases, whether two painted faces are parallel or not.

Case 1: no painted faces are parallel, so that the three painted faces have a common vertex. Removing the painted cubes leaves a $3 \times 3 \times 3$ cube, so the number of unpainted unit cubes is $3^3 = 27$ unit cubes.

Case 2: a pair of painted faces are parallel. Removing the painted cubes leaves a block that is $2 \times 3 \times 4 = 24$ unit cubes.

These two cases are shown, with the painted cubes (blue paint) partially separated from the block of unpainted cubes.



So the number of unpainted cubes is either 27 or 24. However, only 24 is one of the choices in the question,

hence (C).

24. *Alternative 1*

The probability that they both roll the same is $\frac{1}{6}$, because whatever number Ben rolls there is a 1 in 6 chance that Jerry rolls the same. So the probability they roll different scores is $\frac{5}{6}$.

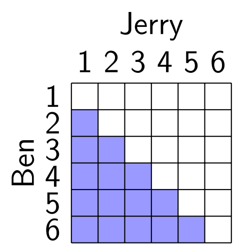
Half of those times Ben will have the higher score, therefore $\frac{5}{12}$,

hence (C).

Alternative 2

Draw a 6×6 grid representing the 36 possibilities for both rolls, then shade those squares where Ben wins. There are 15 such squares.

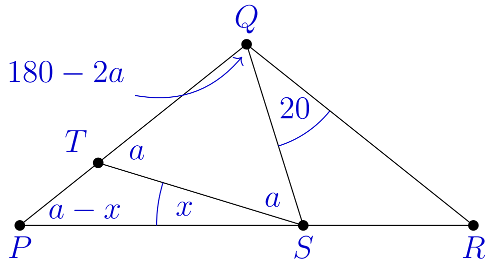
Since all 36 pairs are equally likely, the probability that Ben wins is $\frac{15}{36} = \frac{5}{12}$,



hence (C).

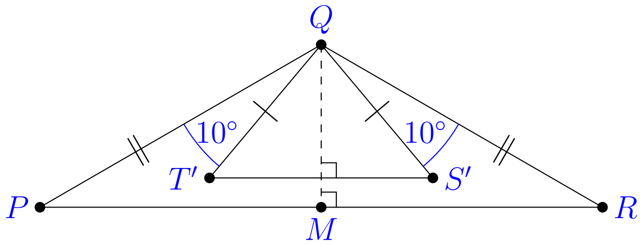
25. *Alternative 1*

Let $\angle PST = x$ and $\angle QTS = \angle QST = a$.
Then $\angle TPS = a - x$.
Then from the exterior angle of $\triangle QRS$, we get $a + x = 20 + a - x$
So $2x = 20$ and $x = 10$,



hence (A).

Alternative 2
The two isosceles triangles $\triangle QPR$ and $\triangle QTS$ each have a line of symmetry through Q. This solution first aligns them. Rotate the $\triangle QTS$ by 10° about point Q to get $\triangle QT'S'$ with $\angle PQT' = \angle RQS' = 10^\circ$.

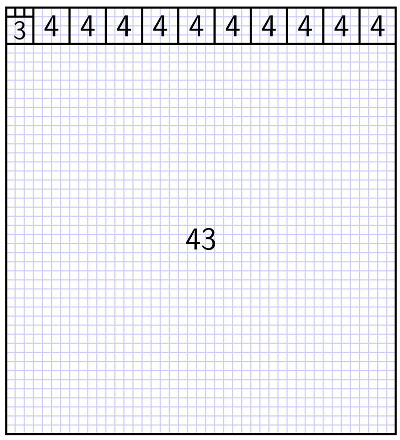


Then the diagram is symmetric about QM, the angle bisector of $\angle PQR$, and also of $\angle TQS$. Thus $PR \perp QM$ and $S'T' \perp QM$ so that $PR \| S'T'$. Since the angle between ST and $S'T'$ is 10°, the angle between ST and PR is 10°,

hence (A).

26. *Alternative 1*

Following instructions, the 43×47 rectangle is cut into a 43×43 square and a 43×4 rectangle.
Then the 43×4 rectangle has ten 4×4 squares cut off, leaving a 3×4 rectangle.
Then the 3×4 square has a 3×3 square cut off, leaving a 3×1 rectangle.
This is cut into into three 1×1 squares.
In total:



Side	Perimeter	Count	Total
43	172	1	172
4	16	10	160
3	12	1	12
1	4	3	12
			356

Then the combined perimeter is 356 units,

hence (356).

Alternative 2

Working backwards, if the final square is $a \times a$, then the size (side length) of each square is a multiple of a. Hence the width and the height of every rectangle is a multiple of a. In particular, a is a divisor of both 43 and 47. However, 43 and 47 are both prime, so $\gcd(43, 47) = 1$, and $a = 1$.

As we work from the 47×43 rectangle to the final 1×1 square, each cut reduces either the width or the height of the rectangle by the size of the square that is cut off. Since the width is reduced by $47 - 1 = 46$ and the height is reduced by $43 - 1 = 42$, the sum of the sizes of all removed squares is 88. Including the 1×1 square left at the end, the sum of the sizes of all squares is 89. The sum of perimeters is then $4 \times 89 = 356$,

hence (356).

Note: The process of cutting off squares is a geometric representation of the Euclidean algorithm for finding the greatest common divisor of two numbers. The argument in this solution then generalises to an $m \times n$ rectangle, where the total perimeter of all squares is $4(m + n - \gcd(m, n))$.

27. Consider first choosing the two chairs in order. There are 14 choices of the first chair. After that choice, there are 12 choices of the second chair, since both the first chair and the chair opposite can't be chosen. So there are $14 \times 12 = 168$ ways of choosing the two chairs in order. However, each pair of chairs is counted twice by this method, so the number of ways of choosing a pair that are not opposite each other is $168 \div 2 = 84$,

hence (84).

28. *Alternative 1*

Whatever the freestyle time is, the breaststroke time is 3 times longer, the butterfly 1.5 times longer and backstroke 2 times longer. So the complete race is 7.5 times the freestyle leg.

Since the full medley is 360 seconds, the freestyle leg is $360 \div 7.5 = 48$ seconds, and we can work out each other leg as follows:

Leg	butterfly	backstroke	breaststroke	freestyle
Time (seconds)	72	96	144	48

Then 240 seconds into the race, I have swum 100 metres of butterfly in 72 seconds, 100 metres of backstroke in 96 seconds and breaststroke for $240 - 72 - 96 = 72$ seconds. This is half-way through the backstroke leg, so I will have swum 50 metres of breaststroke. In all, I have swum $100 + 100 + 50 = 250$ metres after 4 minutes,

hence (250).

Alternative 2

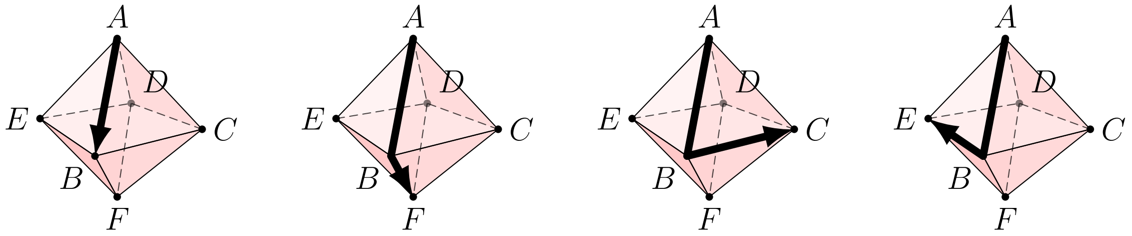
Suppose my freestyle time is x minutes. Then my breaststroke time is $3x$ minutes, my butterfly time is $1.5x$ minutes and my backstroke time is $2x$ minutes. My total time for the relay is $7.5x$ minutes.

Then $7.5x = 6 \implies x = \dfrac{6}{7.5} = 0.8$. Consequently the times for butterfly, backstroke, breaststroke and freestyle are $1.2 + 1.6 + 2.4 + 0.8 = 6$ minutes.

Considering the first 4 minutes, note that $4 = 1.2 + 1.6 + 1.2$ so that after 4 minutes I have swum 100 m of butterfly, 100 m of backstroke and 50 m of breaststroke, a total distance of $100 + 100 + 50 = 250$ metres,

hence (250).

29. Label the vertices A, B, C, D, E and F, and consider first the paths that start AB, as shown in the first diagram. The third vertex is F, C or E.



If the third vertex is F, then the rest of the path is either CDE or EDC, so there are two possible paths.

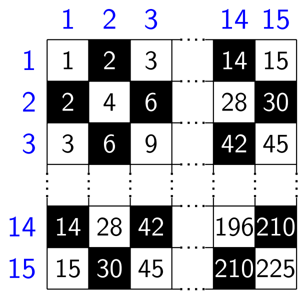
If the third vertex is C, then the rest of the path is one of DEF, DFE, FDE or FED, which is four possible paths. By symmetry, if the third vertex is E there are also four possible paths.

Thus there are $2 + 4 + 4 = 10$ paths that start AB.

Due to the symmetry of the octahedron, there will be 10 paths that start AC, 10 that start AD and 10 that start AE. In all there are $4 \times 10 = 40$ paths,

hence (40).

30. *Alternative 1*



The pattern of white and black squares is as shown. The white squares are those with odd row and odd column along with those with even row and even column.

Consider first the *odd-odd* sum of the numbers with odd row and odd column. We add one row at a time. If k is odd, the sum of the numbers in row k with odd column is

$$\begin{aligned} &k{\times}1 + k{\times}3 + k{\times}5 + k{\times}7 + k{\times}9 + k{\times}11 + k{\times}13 + k{\times}15 \\ &= k(1 + 3 + 5 + 7 + 9 + 11 + 13 + 15) = 64k \end{aligned}$$

The odd-odd sum is then found by adding $64k$ for $k = 1, 3, \ldots, 15$:

$$\begin{aligned} &64{\times}1 + 64{\times}3 + 64{\times}5 + 64{\times}7 + 64{\times}9 + 64{\times}11 + 64{\times}13 + 64{\times}15 \\ &= 64 \times (1 + 3 + 5 + 7 + 9 + 11 + 13 + 15) = 64^2 = 4096 \end{aligned}$$

Similarly, consider the *even-even* sum. For even k, row k includes

$$k(2 + 4 + 6 + 8 + 10 + 12 + 14) = 56k$$

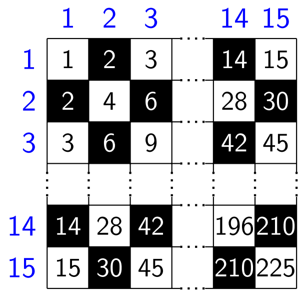
and the sum across all even rows is then

$$56(2 + 4 + 6 + 8 + 10 + 12 + 14) = 56^2 = 3136$$

Finally, the sum of the numbers in all white squares is $4096 + 3136 = 7232$,

hence (232).

Alternative 2



The pattern of white and black squares is as shown. In the first two rows, the white squares can be added:

$$(1+15)+(3+13)+(5+11)+(7+9) = 4\times 16 = 64$$
$$(4+28)+(8+24)+(12+20)+16 = 3\times 32+16 = 112$$

In row 3, the white squares' total is 3 times the total in row 1, so 3×64. Similarly for odd $k = 5, 7, \ldots, 15$, the white squares add to $64k$.

Similarly, in even rows $k = 4, 6, \ldots, 14$, the white squares add to $112\times\dfrac{k}{2}$.

Over the whole grid, the white squares add to

$$\begin{aligned}
&64+64\times 3+64\times 5+\cdots+64\times 15 \ + \ 112+112\times 2+\cdots+112\times 7\\
&= 64(1+3+\cdots+15)+112(1+2+\cdots+7)\\
&= 64\times 64+112\times 28\\
&= 7232
\end{aligned}$$

The last three digits of this total are 232,

hence (232).

Intermediate Solutions

1. There are 10 edges, each 2 cm long, so the total perimeter is $2 \times 10 = 20$ cm,

hence (D).

2. $2000 - 200 + 20 - 2 = 1800 + 20 - 2 = 1820 - 2 = 1818$,

hence (C).

3. The angle marked $a°$ is cointerior to the 145° angle. Since cointerior angles add to 180°, $a = 180 - 145 = 35$,

hence (A).

4. 50% of $\frac{1}{2}$ is $\frac{1}{4}$. So 50% more than $\frac{1}{2}$ is $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$,

hence (D).

5. $\dfrac{1+3+5+7+9}{2+4+6+8+10} = \dfrac{25}{30} = \dfrac{5}{6}$,

hence (B).

6. The shaded area is one-quarter of the square's area. So the square has area 64 square units and side 8 units,

hence (B).

7. The midpoint corresponds to the average of 100 and 10000, which is $\frac{1}{2}(100+10000) = 5050$,

hence (E).

8. (Also J11)

Alternative 1

There are four bags, and one-fifth of a bag is lucerne. Let the lucerne be one unit, then one bag has 5 units and four bags has 20 units. So the lucerne is 1 part in 20 in the total mix, which is 5%,

hence (B).

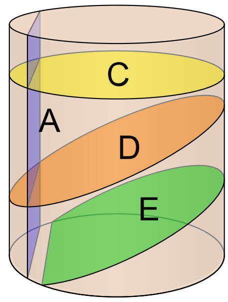
Alternative 2

Each bag is one-quarter or 25% of the mix. The lucerne is one-fifth of one bag, which is 5% of the mix,

hence (B).

9. Consider possible faces from cuts at different angles. These are

- a circle, if the cut is horizontal
- a rectangle, if the cut is vertical
- an oval (ellipse) possibly with one or two segments removed, if the cut is slanted



Looking at the shapes in the question, the triangle (B) is the only shape that isn't one of these shapes. Checking, shapes (A), (C), (D) and (E) can be found as shown. Therefore the triangle is the only shape that can't be the shape of the cut face,

hence (B).

10. *Alternative 1*

Diya took $18\frac{1}{4} = \frac{73}{4}$ minutes. So her average time per lap was $\frac{73}{4} \div 5 = \frac{73}{20} = 3\frac{13}{20} = 3\frac{39}{60}$. Consequently Diya's average lap time was 3 minutes and 39 seconds,

hence (D).

Alternative 2

We need one-fifth of 18 minutes and 15 seconds. We can regroup this into 15 minutes and 195 seconds. One-fifth of this is 3 minutes and 39 seconds,

hence (D).

11. (Also J17)

Let x be the smallest number, so that the four numbers are $x, x+2, x+4$ and $x+6$.

Then $x+6 = 2x-1$. Solving, $x+7 = 2x$ and then $x = 7$.

Consequently the numbers are $7, 9, 11, 13$ and the largest is 13,

hence (C).

12. *Alternative 1*

Firstly, 5 minutes is 300 seconds, which is stored as $300 \times 44100 = 13230000$ samples.

Then this requires $4 \times 13230000 = 52920000$ bytes of storage.

This is closest to 50 million bytes,

hence (D).

Alternative 2

Each second of audio is stored in $4 \times 44100 = 176400$ bytes, which is 0.1764 megabytes.

Then each minute is stored in $0.1764 \times 60 \approx \frac{1}{6} \times 60 = 10$ megabytes.

So 5 minutes is stored in approximately 50 megabytes, or 50 million bytes,

hence (D).

13. *Alternative 1*

The internal angles of the triangle, in degrees, are x, $180 - 3x$ and $180 - (x+30)$, which must add to 180. Then

$$\begin{aligned} x \ + \ 180 - 3x \ + \ 180 - (x+30) &= 180 \\ -3x &= -150 \\ x &= 50 \end{aligned}$$

hence (C).

Alternative 2

Label the angle supplementary to $3x$ as y. In the triangle, the exterior angle $x+30$ is equal to $x+y$, the sum of the other two interior angles. Thus $y = 30$, $3x = 150$ and $x = 50$,

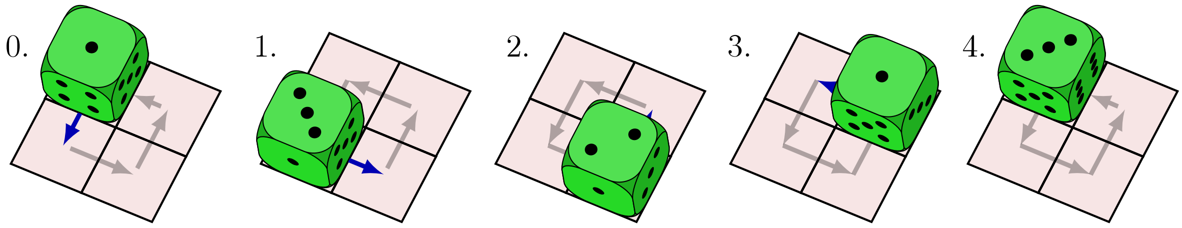
hence (C).

14. To bisect the square the line must pass through its centre, which has coordinates $\left(\frac{3}{2}, \frac{1}{2}\right)$.

Since it also passes through the origin, it has slope $\dfrac{\frac{1}{2}-0}{\frac{3}{2}-0} = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$. Consequently the line has equation $y = \frac{1}{3}x$,

hence (A).

15. (Also J20)
Here is the result of rolling the dice. Note that the corner of the dice in the centre of the 2×2 square remains there, and it is the corner common to the faces with 4, 5, and 6 pips.



Then the upward-facing side is ,

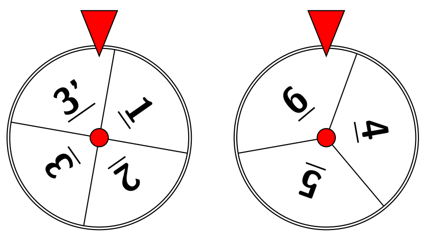
hence (C).

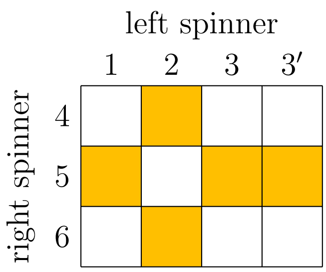
16. *Alternative 1*
An even total results from either an even number on both spinners or an odd number on both spinners. That is

$$\begin{aligned} P(\text{even sum}) &= P(\text{left}{=}2) \times P(\text{right}{=}4 \text{ or } 6) + P(\text{left}{=}1 \text{ or } 3) \times P(\text{right}{=}5) \\ &= \frac{1}{4} \times \frac{2}{3} + \frac{3}{4} \times \frac{1}{3} \\ &= \frac{2+3}{12} = \frac{5}{12} \end{aligned}$$

hence (E).

Alternative 2
The left spinner does not have equally likely outcomes, but it can be made to have 4 equally likely outcomes by splitting the 180° sector in half. Then the sample space for the two spinners has 12 equally likely outcomes, as represented by the grid shown.





The five shaded squares are those where the sum of the two numbers is even, so the probability is $\frac{5}{12}$,

hence (E).

17. The rectangle has area $(a+b)(a+c) = a^2 + ab + ac + bc$. Of the four unshaded triangles, two have area $\frac{1}{2}a^2$ and two have area $\frac{1}{2}bc$.
So the shaded area is $a^2 + ab + ac + bc - a^2 - bc = ab + ac$,

hence (A).

18. *Alternative 1*

Squaring both sides,

$$\begin{aligned}\sqrt{10n+k} &= k\\ 10n+k &= k^2\\ 10n &= k(k-1)\end{aligned}$$

So either k or $k-1$ is a multiple of 5. The smallest such k is then $k=5$.

Hence, $10n = 20$ so $n = 2$. Putting $k = 5$ and $n = 2$ into the original equation gives $\sqrt{25} = 5$, which confirms that we have a solution,

hence (C).

Alternative 2

Since $10n + k$ is a square number greater than 10, $k = \sqrt{10n+k}$ must be at least 4.

If $k = 4$ then $10n + 4 = 16$ and so $n = 1.2$, which is not a solution.

If $k = 5$ then $10n + 5 = 25$ and so $n = 2$. Checking, $\sqrt{10 \times 2 + 5} = \sqrt{25} = 5$, which is a solution, so that $k = 5$ is the smallest possible value of k,

hence (C).

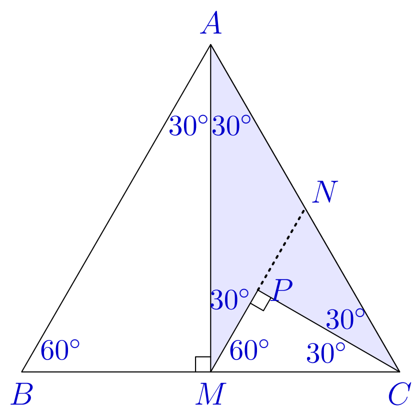
19. Let a be the side length of the larger square and b the side length of the smaller square. Then

$$\begin{aligned}\frac{1}{8}a^2 &= \frac{2}{9}b^2\\ \frac{a^2}{b^2} &= \frac{16}{9}\\ \frac{a}{b} &= \frac{4}{3}\end{aligned}$$

so $a : b = 4 : 3$,

hence (E).

20. *Alternative 1*



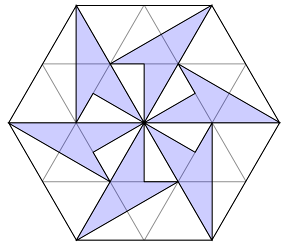
The diagram has rotational symmetry, so we only consider one-sixth of the hexagon, inside an equilateral triangle ABC, which we take to be the unit area. Many angles can be deduced as shown.

Here M is the midpoint of BC so that $\triangle AMC$ has area $\frac{1}{2}$. Also the equilateral triangle $\triangle CMN$ has area $\frac{1}{4}$, and $\triangle CMP$ has area $\frac{1}{8}$. Hence the shaded dart has area $\frac{1}{2} - \frac{1}{8} = \frac{3}{8}$ units.

Over the whole hexagon, each dart shades $\frac{3}{8}$ of the area of an equilateral triangle, so the total shaded area is $\frac{3}{8}$ of the area of the hexagon,

hence (E).

Alternative 2



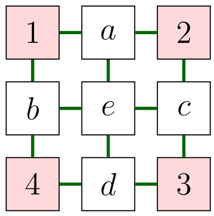
First draw a grid of equilateral triangles over the hexagon, where the unit is half the hexagon's side length. Then many of the angles and points in the darts align with the grid, or are at 30° to the grid.

In particular, each dart is 3 half-units of area, so that there are $6 \times \dfrac{3}{2} = 9$ units shaded out of 24. So the shaded area is $\dfrac{9}{24} = \dfrac{3}{8}$ of the area of the hexagon,

hence (E).

21. *Alternative 1*

Label the empty squares as shown.



Since each labelled square is an average of its neighbours, we have 5 equations:

$$
\begin{aligned}
3 + e &= 3a \\
5 + e &= 3b \\
5 + e &= 3c \\
7 + e &= 3d \\
a + b + c + d &= 4e
\end{aligned}
$$

Adding the first 4 equations,

$$20 + 4e = 3(a + b + c + d) = 12e$$

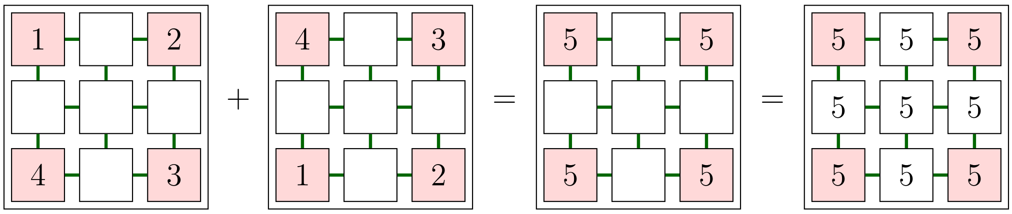
Then $20 = 8e$, so $e = \dfrac{5}{2}$ and $a = \dfrac{1}{3}(3 + e) = \dfrac{11}{6}$,

hence (E).

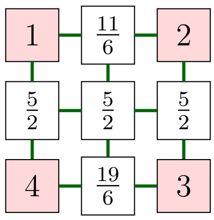
Alternative 2

A remarkable property of grids with these averaging properties is that when two such grids are added (square by square), the *sum grid* has the same averaging properties.

So when the grid in the problem is added to the same grid that has been flipped top to bottom, then the sum grid will also have the same averaging properties. This sum grid has 5 in each corner. Since each of the unknown numbers must lie between the smallest and largest values of its neighbours, this is only possible if all values are 5.



Working back, the middle row of the original grid must have $\frac{5}{2}$ in each square, and the last two squares are the averages of 3 known neighbours:



hence (E).

22. If the intended time had only 2 digits, then the difference between the times would be less than 1 minute, and if the intended time had 4 or more digits, then the difference would be at least 9 minutes. Therefore the intended time had 3 digits, 'abc' say. That is, Rick meant to enter 'a minutes bc seconds', but instead entered 'ab seconds'.

In seconds, the intended time is equal to $60a + 10b + c$ and the mistyped time is $10a + b$, so the difference is $50a + 9b + c$. Setting this equal to 4 minutes 42 seconds, we have

$$50a + 9b + c = 282$$

For $a{:}bc$ to be a valid time, a and c are at most 9, and b is at most 5. In particular, $9b+c$ is at most $9\times5+9 = 54$, so we must have $50a = 250$. Then $a = 5$ and $9b+c = 282-250 = 32$. Given the possible values of digits b and c, the only solution is $b = 3$ and $c = 5$. That is, Rick meant to type 5 minutes 35 seconds but instead typed 53 seconds, so the missing digit was a 5,

hence (C).

23. Suppose the original cube is $n \times n \times n$ unit cubes. Since 288 is between $6^3 = 216$ and $7^3 = 343$, n is at least 7. On the other hand, for $n = 9$ or more, there are at least $7^3 = 343$ unit cubes that are unpainted due to not being on the surface. This is too many, so $n = 7$ or $n = 8$.

For each painted face, suppose we remove all unit cubes with paint from that face. This amounts to removing the layer of unit cubes adjacent to the face, reducing one dimension of the large cube by 1. Once this is done for a number of painted faces, a block $a \times b \times c$ is obtained, where $abc = 288$, and each of a, b, c is n, $n-1$ or $n-2$.

Since n is 7 or 8, a, b and c are chosen from 5, 6, 7 or 8. Of these, only 6 and 8 are factors of 288, which is not a perfect cube. So abc is either $6 \times 6 \times 8 = 288$ (yes) or $6 \times 8 \times 8 = 384$ (no).

This $6 \times 6 \times 8$ block is made by reducing the $8 \times 8 \times 8$ cube by 2 units in each of 2 dimensions. This can only be from the original cube having 4 painted faces,

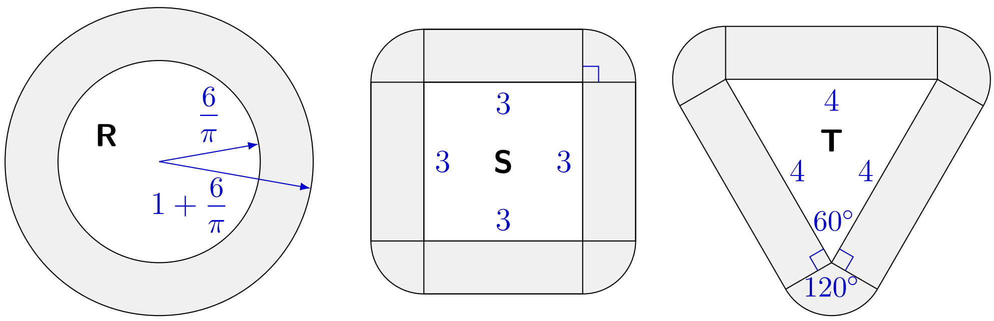
hence (D).

24. Simplifying a product of fractions is easier when each factor has numerator and denominator fully factorised. So we write each factor as $1 - \frac{1}{n^2} = \frac{n^2-1}{n^2} = \frac{(n-1)}{n}\frac{(n+1)}{n}$. Then the product is

$$\begin{aligned}
&\left(\frac{1}{2}\times\frac{3}{2}\right)\times\left(\frac{2}{3}\times\frac{4}{3}\right)\times\left(\frac{3}{4}\times\frac{5}{4}\right)\times\left(\frac{4}{5}\times\frac{6}{5}\right)\times\cdots\times\left(\frac{13}{14}\times\frac{15}{14}\right)\times\left(\frac{14}{15}\times\frac{16}{15}\right)\\
&\qquad=\frac{1}{2}\times\left(\frac{3}{2}\times\frac{2}{3}\right)\times\left(\frac{4}{3}\times\frac{3}{4}\right)\times\left(\frac{5}{4}\times\frac{4}{5}\right)\times\cdots\times\left(\frac{15}{14}\times\frac{14}{15}\right)\times\frac{16}{15}\\
&\qquad=\frac{1}{2}\times 1\times 1\times 1\times\cdots\times 1\times\frac{16}{15} = \frac{8}{15}
\end{aligned}$$

hence (B).

25. Each fishing zone can be constructed from circles, sectors and rectangles. Since each sector's area can be calculated as a fraction of a full circle, we can work out each area.



The fishing zone for Razz has area $\pi\left(1+\frac{6}{\pi}\right)^2 - \pi\left(\frac{6}{\pi}\right)^2 = \pi + 12 + \frac{36}{\pi} - \frac{36}{\pi} = \pi + 12.$

The fishing zone for Sazz has area $4 \times (3 \times 1) + 4 \times \left(\frac{\pi}{4}\right) = \pi + 12.$

The fishing zone for Tazz has area $3 \times (4 \times 1) + 3 \times \left(\frac{\pi}{3}\right) = \pi + 12.$

So all three fishing zones have area $\pi + 12$ square kilometres,

hence (E).

Note: This illustrates a more general result. For any convex shape A of perimeter P, the area of the region B within r units of P is $Pr + \pi r^2$. To see this when A is a polygon, divide B into rectangles and sectors similar to shapes S and T above. For more general A, the same conclusion can be reached by considering polygons with an increasing number of sides that approximate A with increasing accuracy.

26. *Alternative 1*

The number of goals (sixes) is 0, 1, 2 or 3.

With 0 sixes and 20 ones, there is one possible scoring sequence: $(1, 1, \ldots, 1)$.

With 1 six and 14 ones, there are 15 scoring sequences, depending on where the goal occurs in the list: $(6, 1, 1, \ldots, 1)$, $(1, 6, 1, \ldots, 1)$, $\ldots$, $(1, 1, 1, \ldots, 6)$.

With 2 sixes and 8 ones, there are 45 scoring sequences. One way to calculate this is to consider choosing 2 items from 10, for which there are $\dfrac{10 \times 9}{2 \times 1} = 45$ ways.

Alternatively, the first six can occur in any of 9 positions, but the number of possibilities for the second six depends on where the first six is. If the first six is in position 1, there are 9 possibilities for the second. If the first is in position 2, there are 8 possibilites for the second, and so on. In all there are $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45$ possibilities.

With 3 sixes and 2 ones, there are 10 scoring sequences. This can be calculated similarly to the previous case, either as $\dfrac{5 \times 4}{2}$ or as $4 + 3 + 2 + 1$.

In all, there are $1 + 15 + 45 + 10 = 71$ possibilities,

hence (71).

Alternative 2

Make a table of the number of sequences that score n points for $n = 1, 2, 3, \ldots$. Clearly, there is only one way for each n up to 5, and two ways for $n = 6$.

For each $n > 6$, the total is made up of two types of sequences: those ending in 6 and those ending in 1. When the sequence ends in 6, the number of ways is the same as the number of ways of scoring $n - 6$ points. When the sequence ends in 1, the number of ways is the same as the number of ways of scoring $n - 1$ points. In total, the number of ways of scoring n points is equal to the number of ways of scoring $n - 6$ points plus the number of ways of scoring $n - 1$ points.

Using this rule, we can fill in the table up to $n = 20$:

Points	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Sequences	1	1	1	1	1	2	3	4	5	6	7	9	12	16	21	27	34	43	55	71

Consequently there are 71 sequences with a final score of 20,

hence (71).

27. *Alternative 1*

Let $x = 100004$ and $P = (x-4)(x-2)(x+2)(x+4)$. Then

$$\begin{aligned} N = P + n &= (x-4)(x-2)(x+2)(x+4) + n \\ &= (x^2-4)(x^2-16) + n \\ &= x^4 - 20x^2 + 64 + n \\ &= (x^2-10)^2 - 36 + n \end{aligned}$$

Let $k = x^2 - 10 = 100004^2 - 10$, so that when $n = 36$, $N = k^2$, a perfect square.

To show that this is the smallest n, we only need show that the preceding square number $(k-1)^2$ is less than or equal to P. This is certainly true, since $P = k^2 - 36$ whereas $(k-1)^2 = k^2 - (2k+1) = P + 36 - (2k+1)$ which is less than P, since $k \approx 10^{10}$.

Then $(k-1)^2 = k^2 - 2k + 1 = P + 37 - 2k$. But $k \approx 10^{10}$, so $(k-1)^2 \approx P + 37 - (2 \times 10^{10})$, which is less than P.

Consequently, $n = 36$ is the smallest natural number solution,

hence (36).

Alternative 2

Let $a = 10^5$ and $P = a(a+2)(a+6)(a+8)$. We want $N = P + n$ to be the smallest perfect square greater than P. Then

$$P = a(a+8) \times (a+2)(a+6) = (a^2+8a)(a^2+8a+12)$$

Let $b = a^2 + 8a = 10\,000\,800\,000$, then

$$P = b(b+12) = b^2 + 12b = (b+6)^2 - 36$$

Then $P + 36 = (b+6)^2 = 10\,000\,800\,006^2$, so that $n = 36$ is a value that makes $N = P + n$ into a perfect square.

To check that $(b+6)^2$ is the smallest perfect square greater than P, consider $(b+5)^2$, the preceding perfect square:

$$P - (b+5)^2 = b^2 + 12b - (b+5)^2 = 2b - 25 \approx 2 \times 10^{10}$$

so that $(b+5)^2$ is (much) smaller than P.

Thus $n = 36$ is the smallest natural number for which $P + n$ is a perfect square,

hence (36).

28. We separately consider cases based on the choice of colours and matchsticks, then work out possible arrangements of the chosen matchsticks.

Case 1: All four matches the same colour. There are 4 ways to choose the colour. For each choice, there is only one square. So there are 4 possibilities for case 1.

Case 2: Two colours with 3 matches of one and 1 of the other. There are 4 ways to choose the 3-match colour and 3 ways to choose the 1-match colour. Once chosen, all squares are the same by rotation. So there are $4 \times 3 = 12$ possibilities for case 2.

Case 3: Two colours with 2 matches of one and 2 of the other. There are 6 ways to choose two colours from 4 (RY, RB, RG, YB, YG, BG). Once chosen there are two different arrangements depending on whether like colours are opposite or adjacent. For example, if the chosen colours are RY, then the two arrangements are RYRY and RRYY, in order around the square. So there are $6 \times 2 = 12$ possibilities for case 3.

Case 4: Three colours with two matches of one colour, and one match of each of the other two colours. There are 4 ways to choose the 2-match colour, and then 3 ways to choose the other two colours. For instance, if R is chosen as the 2-match colour, then the other two matches can be YB, YG or BG. Once chosen, there are two different arrangements depending on whether the two like colours are opposite or adjacent. For example, with two R's, Y and B, arrangements are RYRB and RRYB. So there are $4 \times 3 \times 2 = 24$ possibilities for case 4.

Case 5: Four colours. There is only one way to choose all four colours: RYBG. However, there are 3 arrangements, based on which colour is opposite Red. These can be written RYBG, RBYG and RYGB. So there are 3 possibilities for case 5.

The total number of different squares that can be made is $4 + 12 + 12 + 24 + 3 = 55$,

hence (55).

29. *Alternative 1*

For a divisor n, let r be the remainder in $499 \div n$ and s be the remainder in $500 \div n$. Consider a table of these remainders. To assist in finding $M - N$, tabulate $r - s$:

n	1	2	3	4	5	6	7	8	9	10	$\cdots$	499	500	sum
r	0	1	1	3	4	1	2	3	4	9	$\cdots$	0	—	M
s	0	0	2	0	0	2	3	4	5	0	$\cdots$	1	0	N
$r - s$	0	1	-1	3	4	-1	-1	-1	-1	9	$\cdots$	-1	—	$M - N$

The column $n = 500$ is not included in the sum M and contributes 0 to the sum N, so we remove it.

Claim: When n is a divisor of 500, $r - s = n - 1$. Otherwise $r - s = -1$.

When n is a divisor of 500, say $nm = 500$, then $500 \div n = m$ exactly, so $s = 0$. Also $499 \div n = (m - 1) + \dfrac{n-1}{n}$ so that $r = n - 1$. Then $r - s = n - 1$. We can sum the 11 cases where n is a divisor of 500 separately:

n	1	2	4	5	10	20	25	50	100	125	250	sum
r	0	1	3	4	9	19	24	49	99	124	249	581
s	0	0	0	0	0	0	0	0	0	0	0	0
$r - s$	0	1	3	4	9	19	24	49	99	124	249	581

Otherwise, $r - s = 1$, since when n is not a divisor of 500, the remainder in $500 \div n$ is one more than the remainder in $499 \div n$. Since there are 11 proper divisors of 500, the number of columns where $r - s = -1$ is $499 - 11 = 488$, and so they contribute -488 to $M - N$.

In total, $M - N = 581 - 488 = 93$,

hence (93).

Alternative 2

Let Bluey's sequence be $m_1, m_2, \ldots, m_{499}$ and let Jean-Luc's sequence be $n_1, n_2, \ldots, n_{500}$. We start by making the following observations, using the notation $a \mid b$ to mean that a divides b:

(i) For $i \mid 500$ and $1 \leqslant i \leqslant 499$, $m_i = i - 1$ and $n_i = 0$.

(ii) For $i \nmid 500$ and $1 \leqslant i \leqslant 499$, $m_i + 1 = n_i$.

So we can calculate $M - N$ as follows:

$$\begin{aligned}
M - N &= \sum_{i=1}^{499} m_i - \sum_{i=1}^{500} n_i \\
&= \sum_{i=1}^{499} (m_i - n_i) && \text{(Since } n_{500} = 0.\text{)} \\
&= \sum_{\substack{i \mid 500 \\ 1 \leqslant i \leqslant 499}} (m_i - n_i) \quad + \sum_{\substack{i \nmid 500 \\ 1 \leqslant i \leqslant 499}} (m_i - n_i) \\
&= \sum_{\substack{i \mid 500 \\ 1 \leqslant i \leqslant 499}} (i - 1) \quad + \sum_{\substack{i \nmid 500 \\ 1 \leqslant i \leqslant 499}} (-1) && \text{(Observations (i) and (ii).)} \\
&= \sum_{\substack{i \mid 500 \\ 1 \leqslant i \leqslant 499}} i \quad + \sum_{i=1}^{499} (-1) \\
&= 1 + 2 + 4 + 5 + 10 + 20 + 25 + 50 + 100 + 125 + 250 \ - \ 499 \\
&= 93
\end{aligned}$$

hence (93).

30. Let the dimensions of the large rectangle be x tiles by y tiles. Then xy tiles are used and $2x + 2y - 4$ of these form the perimeter. As this is $\frac{1}{5}$ of the total, we have $xy = 5(2x + 2y - 4) = 10x + 10y - 20$. Then

$$\begin{aligned}
xy - 10x - 10y + 20 &= 0 \\
xy - 10x - 10y + 100 &= 80 \\
(x - 10)(y - 10) &= 80
\end{aligned}$$

Hence $x - 10$ and $y - 10$ are integer factors of 80.

The possibilities are (1,80), (2,40), (4,20), (5,16) and (8,10). These give areas of $11 \times 90 = 990$, $12 \times 50 = 600$, $14 \times 30 = 420$, $15 \times 26 = 390$ and $18 \times 20 = 360$ respectively.

There are also factorisations of 80 into pairs of negative factors that could provide values for $x - 10$ and $y - 10$, such as $80 = (-16) \times (-5)$. However for these, one of the factors (say $x - 10$) is -10 or less, so x is not positive. So these negative factorisations provide no solutions.

Noticing that $990 = 600 + 390$ gives us the dimensions of our three rectangles. So there are 990 tiles, of which $990 \times \frac{4}{5} = 792$ are blue,

hence (792).

Senior Solutions

1. Of the 16 small triangles, 6 are shaded. As a fraction, $\dfrac{6}{16} = \dfrac{3}{8}$,

hence (E).

2. $2021 \div 7 = 288\,\text{remainder}\,5$,

hence (D).

3. $12 - \dfrac{1}{12} - 1 = 11 - \dfrac{1}{12} = 10\dfrac{11}{12}$,

hence (C).

4. The angles in a quadrilateral sum to 360°, so that $3\theta + 150 = 360$. Then $3\theta = 210$ so $\theta = 70$,

hence (A).

5. $\dfrac{(20 \times 21) + 21}{21} = \dfrac{21 \times 21}{21} = 21$,

hence (B).

6. The time Henry took in minutes was $3\frac{3}{4} = \dfrac{15}{4}$. In hours, this is $\dfrac{15}{4} \times \dfrac{1}{60} = \dfrac{1}{16}$.

His average speed in kilometres per hour was $\dfrac{3}{2} \div \dfrac{1}{16} = \dfrac{3}{2} \times \dfrac{16}{1} = 24$,

hence (C).

7.

$$\frac{8^3 \times 3^6}{6^5} = \frac{(2^3)^3 \times 3^6}{(2 \times 3)^5} = \frac{2^9 \times 3^6}{2^5 \times 3^5} = 2^{9-5} \times 3^{6-5} = 2^4 \times 3 = 48$$

hence (B).

8. The product of $\dfrac{1}{3} = 0.\dot{3}$ and $\dfrac{2}{3} = 0.\dot{6}$ is $\dfrac{1}{3} \times \dfrac{2}{3} = \dfrac{2}{9}$. As a decimal this is $0.\dot{2}$,

hence (B).

9. The area of the parallelogram is the base times the height perpendicular to the base. From the coordinates given, the base is parallel to the x-axis, of length $(a - 1)$. The height is then parallel to the y-axis, of length $8 - 2 = 6$. Hence $6(a - 1) = 48$, so $a - 1 = 8$ and $a = 9$,

hence (C).

10. Let the four walls be north, south, east and west.

There are three choices for the ceiling colour.

Then the north wall must be a second colour, so there are two choices for the north wall.

Then there is no further choice: the east and west walls will be the third colour, and the south wall the second colour.

In all there are $3 \times 2 = 6$ ways that the room can be painted,

hence (C).

11. Fractions between $\frac{1}{5}$ and $\frac{4}{5}$ with denominator less than 6 are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{2}{5}$ and $\frac{3}{5}$. In ascending order, these are

$$\frac{1}{5}, \boxed{\frac{1}{4}}, \frac{1}{3}, \frac{2}{5}, \boxed{\frac{1}{2}}, \boxed{\frac{3}{5}}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$$

Thus, the sum of the three missing fractions is $\frac{1}{4} + \frac{1}{2} + \frac{3}{5} = \frac{5+10+12}{20} = \frac{27}{20}$,

hence (D).

12. *Alternative 1*

Tabulate the number of each after each week.

End of week	0	1	2	3	4	5
Flowers	150000	75000	37500	18750	9375	4688
Fungi	20	60	180	540	1620	4860

At the end of 5 weeks the fungi just outnumber the flowers,

hence (B).

Alternative 2

The ratio of flowers to fungi is initially $7500:1$. Each week the ratio reduces by a factor of 6. We have $6^4 = 1296$ and $6^5 = 7776$. So after 5 weeks the ratio drops below $1:1$,

hence (B).

13. *Alternative 1*

Rearranging,

$$V = \frac{qrB}{m}$$

After the changes described, there is a new value of V:

$$V_{\text{new}} = \frac{(3q)\,rB}{\frac{m}{2}} = \frac{3qrB}{1} \times \frac{2}{m} = \frac{6\,qrB}{m}$$

This is 6 times the initial value of V,

hence (A).

Alternative 2

Rearranging,

$$V = \frac{qrB}{m} = q \times \frac{1}{m} \times rB$$

Halving m doubles $\frac{1}{m}$, so if q is trebled, m is halved and rB is constant, then V is multiplied by 6,

hence (A).

Alternative 3

Use subscripts 1 and 2 to represent initial and final values of q, m and V. Then

$$r = \frac{m_1 V_1}{q_1 B} = \frac{m_2 V_2}{q_2 B}$$

$$\implies \frac{V_2}{V_1} = \frac{m_1}{m_2}\frac{q_2}{q_1} = 2 \times 3 = 6$$

hence (A).

14. If $CD = CE = x$, then $AC = BC = 12 - x$ and $BE = 12 - 2x$. The area of $\triangle BDE$ is $\frac{1}{2}BE \times CD = 9$, so

$$(6 - x)x = 9 \implies x^2 - 6x + 9 = 0 \implies (x - 3)^2 = 0 \implies x = 3$$

Then $BC = 12 - x = 9$ so that area of $\triangle ABD$ is $\frac{1}{2} \times 12 \times 9 = 54$,

hence (C).

15. Write each as a tenth power: $1^{40} = 1^{10}$, $2^{30} = 8^{10}$, $3^{20} = 9^{10}$ and $4^{10} = 4^{10}$. Then $1^{10} < 4^{10} < 8^{10} < 9^{10}$ so $1^{40} < 4^{10} < 2^{30} < 3^{20}$,

hence (A).

16. *Alternative 1*

From the areas of the triangles, $xz = 24$ and $yz = 42$. Eliminating z, it follows that $7x = 4y$. As a triple, $(x, y, z) = (4, 7, 6)$.

The only other integer solutions will occur when $z = 1, 2$ or 3. That is, $(x, y, z) = (24, 42, 1)$ or $(12, 21, 2)$ or $(8, 14, 3)$.

In all, there are 4 solutions,

hence (D).

Alternative 2

Since $\dfrac{1}{2}xz = 12$, we have $xz = 24$. Similarly $yz = 42$. Then z is a common factor of 24 and 42, and hence a factor of $\gcd(24, 42) = 6$. Moreover, if z is any factor of 6, then $x = \dfrac{24}{z}$ and $y = \dfrac{42}{z}$ are integer lengths. The possible values for z are $\{1, 2, 3, 6\}$,

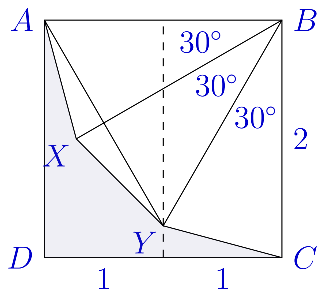
hence (D).

17. *Alternative 1*

Let the square have side length 1. The angle at the apex of each isosceles triangle is $30°$ so, by the area formula $\dfrac{1}{2}ab\sin C$, the area of each triangle is $\dfrac{1}{2} \times 1 \times 1 \times \sin 30° = \dfrac{1}{4}$. Thus the area of the pentagon is $1 - \dfrac{3}{4} = \dfrac{1}{4}$, which is $\dfrac{1}{4}$ of the square's area,

hence (B).

Alternative 2



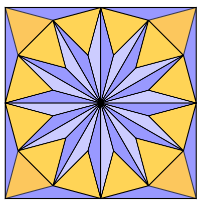
Choose a length unit equal to half the square's side.

Label as shown. In $\triangle ABY$, $\angle B = 60°$ and $AB = BY = 2$ so $\triangle ABY$ is equilateral. Then Y lies on the perpendicular bisector of AB, which also bisects the square.

Hence $\triangle BCY$ has base $BC = 2$, height 1 and area 1. Then the shaded pentagon $AXYCD$ has area $4 - 3 \times 1 = 1$, which is $\dfrac{1}{4}$ of the square's area,

hence (B).

Note: The figure in the question can be developed into this dissection of a square into isosceles and equilateral triangles. The dissection demonstrates that the regular dodecagon inscribed in the square occupies exactly $\dfrac{3}{4}$ of its area. The same dissection can be used to answer this question.



18. The 36 equally likely possibilities from rolling the dice twice can be represented in this 6×6 table.

As shown, 8 of the 36 outcomes have two faces adding to one of the other faces. So the probability is $\frac{8}{36} = \frac{2}{9}$,

	2	3	5	8	13	21
2		•				
3	•		•			
5		•		•		
8			•		•	
13				•		
21						

hence (E).

19. *Alternative 1*

We calculate as follows, using the fact that a and b are positive numbers:

$$\begin{aligned}
\sqrt{a^2 + \frac{1}{b^2}} \times \sqrt{b^2 + \frac{1}{a^2}} &= \sqrt{\left(a^2 + \frac{1}{b^2}\right) \times \left(b^2 + \frac{1}{a^2}\right)} \\
&= \sqrt{a^2b^2 + 2 + \frac{1}{a^2b^2}} \\
&= \sqrt{\left(ab + \frac{1}{ab}\right)^2} \\
&= ab + \frac{1}{ab}
\end{aligned}$$

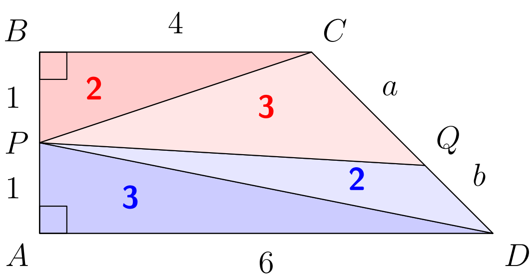
hence (E).

Alternative 2

Since $b > 0$, the first surd is $\sqrt{\frac{a^2b^2+1}{b^2}} = \frac{1}{b}\sqrt{a^2b^2+1}$. Similarly, the second surd is $\frac{1}{a}\sqrt{a^2b^2+1}$. Hence $\sqrt{a^2 + \frac{1}{b^2}} \times \sqrt{b^2 + \frac{1}{a^2}} = \frac{1}{ab}(a^2b^2+1) = ab + \frac{1}{ab}$,

hence (E).

20. The trapezium has area 10, so each half has area 5. Further, some triangles have known areas, so others can be deduced:



Consider $\triangle QCP$ and $\triangle DQP$ as having bases a and b on line DC, with a common altitude. Then the ratio of their areas is $a : b = 3 : 2$,

hence (E).

21. *Alternative 1*

By completing the square twice, the equation becomes $(x+y)^2+(y+1)^2=1989$.

Since the right-hand side is divisible by 3, each square on the left-hand side must be divisible by 3. Let $x+y=3k$ and $y+1=3\ell$, where k and ℓ are positive integers and $k \geqslant \ell$. Then, $k^2+\ell^2=221$. Since $k^2 \geqslant \ell^2$,we get $221/2 \leqslant k^2 \leqslant 221$. Hence, we consider $k^2=121$, $k^2=144$, $k^2=169$ and $k^2=196$, i.e. $k=11$, $k=12$, $k=13$ and $k=14$. The largest value of $x+y$ is 42 for $k=14$. We just need to check that a solution exists for $k=14$. Indeed, for $k=14$ we get $\ell=5$. Thus, $y=14$ and $x=28$,

hence (C).

Alternative 2

As in the first solution, $(x+y)^2+(y+1)^2=1989$. Write $u=x+y$ and $v=y+1$, where $u^2+v^2=1989$ and u is as large as possible.

Since $44^2=1936$ and $45^2=2025$ we consider values of u from 44 down to check whether $1989-u^2$ is a perfect square:

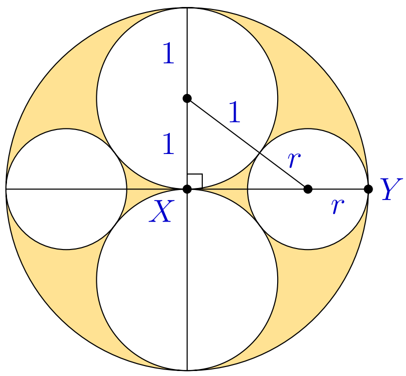
u	44	43	42	41
u^2	1936	1849	1764	
$1989-u^2$	53	140	225	
v	—	—	15	

Then $u=42$, $v=15$ so that $y=14$ and $x=28$, which gives the solution with the largest value of $x+y$,

hence (C).

22. *Alternative 1*

Each of the two larger inscribed circles has radius half the outer circle. So each has one-quarter of the area. To find the area of the two smallest circles, choose a unit length as shown, and let r be the radius of these smallest circles. Then using Pythagoras' theorem



$$\begin{aligned} r+\sqrt{(1+r)^2-1^2} &= XY = 2 \\ \sqrt{r^2+2r} &= 2-r \\ r^2+2r &= r^2-4r+4 \\ 6r &= 4 \\ r &= \frac{2}{3} \end{aligned}$$

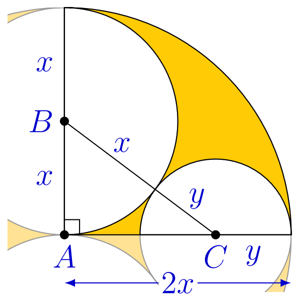
So each of the smallest circles has one-third the radius of the outer circle, hence one-ninth the area. Then as a fraction of the largest circle, the unshaded area is $\frac{1}{4}+\frac{1}{4}+\frac{1}{9}+\frac{1}{9}=\frac{13}{18}$.

So the shaded area is $\frac{5}{18}$ of the diagram,

hence (E).

Alternative 2

Consider the quadrant shown, with radii x, y and $2x$. Then $AC = 2x - y$, so that by Pythagoras' theorem



$$\begin{aligned} x^2 + (2x-y)^2 &= (x+y)^2 \\ x^2 + 4x^2 - 4xy + y^2 &= x^2 + 2xy + y^2 \\ 2x(2x-3y) &= 0 \\ \frac{y}{x} &= \frac{2}{3} \qquad \text{(since } x > 0\text{)} \end{aligned}$$

The shaded fraction of the diagram is

$$\frac{4\pi x^2 - 2\pi x^2 - 2\pi y^2}{4\pi x^2} = 1 - \frac{1}{2} - \frac{1}{2}\left(\frac{y}{x}\right)^2 = \frac{1}{2} - \frac{1}{2}\left(\frac{2}{3}\right)^2 = \frac{5}{18}$$

hence (E).

Alternative 3

Let the three distinct radii in the diagram be 1, $\frac{1}{2}$ and a. Put $k = \frac{1}{a}$, then by Descartes' circle theorem,

$$\begin{aligned} 2\big((-1)^2 + (2)^2 + (2)^2 + k^2\big) &= (-1+2+2+k)^2 \\ 18 + 2k^2 &= (k+3)^2 \\ k^2 - 6k + 9 &= 0 \\ (k-3)^2 &= 0 \end{aligned}$$

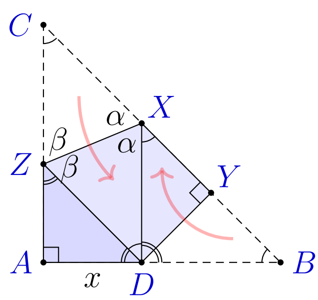
Hence $k = 3$ and $a = \frac{1}{3}$.

So within the outermost circle of area π, the unshaded area is $2\pi\left(\frac{1}{2}\right)^2 + 2\pi\left(\frac{1}{3}\right)^2 = \frac{13}{18}\pi$.

Thus the unshaded area is $\frac{13}{18}$ of the circle, and the shaded area is $\frac{5}{18}$ of the circle,

hence (E).

23. In $\triangle ABC$, $AB = AC = 24$ and $BC = 24\sqrt{2}$. Make both folds along ZX and DY as shown. Known right angles are marked with $\llcorner$, and known 45° angles with $\measuredangle$. Further angles, marked $\measuredangle$, are then easily deduced to be 45°.



Then $\alpha = 135^\circ \div 2 = 67.5^\circ$ and similarly $\beta = 67.5^\circ$. So the two congruent triangles $\triangle DXZ$ and $\triangle CXZ$ are isosceles. In particular, $DX = DZ = CX = CZ$.

By the ASA rule, $\triangle ADZ$ is congruent to $\triangle YDX$, and so it is also congruent to $\triangle YBD$. In particular, $x = AD = ZA = DY = XY = BY$.

Adding $CX = CZ$ to $XY = ZA$, gives $CY = CA = 24$. Finally $x = BY = BC - CY = 24\sqrt{2} - 24 = 24(\sqrt{2} - 1)$,

hence (E).

24. From the question, it is clear that $n \neq 0$, and

$$\frac{a+n}{b+n} = \frac{2a}{b} \quad \text{and} \quad \frac{a-n}{b-n} = \frac{3a}{b}$$
$$ab + bn = 2ab + 2an \quad \text{and} \quad ab - bn = 3ab - 3an$$
$$bn - 2an = ab \quad \text{and} \quad 3an - bn = 2ab$$
$$\begin{aligned} 2bn - 4an = 2ab &= 3an - bn \\ 3bn &= 7an \\ 3b &= 7a && [\text{cancel nonzero } n] \\ a = 3 \text{ and } b &= 7 && [\gcd(a,b) = 1] \\ n &= 21 && [bn - 2an = ab \text{ above}] \end{aligned}$$

Checking, $\frac{3+21}{7+21} = \frac{24}{28} = \frac{6}{7} = 2 \times \frac{3}{7}$ and $\frac{3-21}{7-21} = \frac{-18}{-14} = \frac{9}{7} = 3 \times \frac{3}{7}$,

hence (C).

25. The distances are paired up into three pairs each with the same sum, equal to the side of the cube. Hence $3s = 1+2+3+4+5+6 = 21$ and so $s = 7$.

Pick one face out of the six for distance 1 cm, and then the opposite face must be 6 cm. So there are 6 choices here.

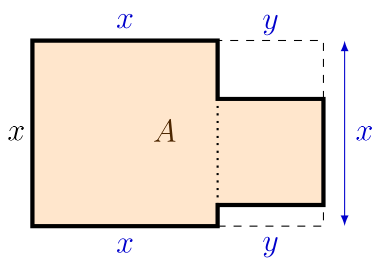
Pick one of the remaining 4 faces for distance 2 cm, and then the opposite face must be 5 cm. There are 4 choices here.

Pick one of the remaining 2 faces for distance 3 cm, and then the opposite face must be 4 cm. There are 2 choices here.

Each of these $6 \times 4 \times 2 = 48$ possibilities gives a different way of having the six distances. So other than the point P, there are 47 other points that have these distances,

hence (D).

26. Let x and y be the side lengths of the squares, and $A = x^2 + y^2$ the combined area, as shown.



The shape's perimeter is the same as that of the bounding rectangle. Thus $4x + 2y = 70$ and so $y = 35 - 2x$. Then

$$\begin{aligned} A &= x^2 + (35 - 2x)^2 \\ &= x^2 + 4x^2 - 140x + 1225 \\ &= 5(x^2 - 28x + 245) \\ &= 5\big((x-14)^2 + 49\big) \end{aligned}$$

So when $x = 14$, $A = 5 \times 49 = 245$ is the smallest value of this quadratic. Checking, this has $y = 35 - 2x = 7$, so $x > y$ meaning that the minimum $A = 245$ occurs within the set of possible values of x,

hence (245).

27. We first solve the problem where m and n need not be distinct.

The prime factorisation of 2310 is $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. So if m is a divisor of 2310 and n is a divisor of m, then $n = 2^a \times 3^b \times 5^c \times 7^d \times 11^e$ and $m = 2^p \times 3^q \times 5^r \times 7^s \times 11^t$ where a, b, c, d, e, p, q, r, s, t are integers with $0 \leqslant a \leqslant p \leqslant 1$, $0 \leqslant b \leqslant q \leqslant 1$, $0 \leqslant c \leqslant r \leqslant 1$, $0 \leqslant d \leqslant s \leqslant 1$, and $0 \leqslant e \leqslant t \leqslant 1$.

Moreover, if a, b, c, d, e, p, q, r, s, t are integers satisfying the above inequalities, then m and n are both divisors of 2310 with n dividing m.

There are three ways to choose a and p: $a = p = 0$, $a = p = 1$ and $a = 0, p = 1$. Similarly there are three ways to choose b and q, three ways to choose c and r, three ways to choose d and s, and three ways to choose e and t. In all there are $3^5 = 243$ ways to choose a, b, c, d, e, p, q, r, s, t.

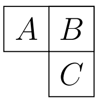
To exclude those where $m = n$, we count these. There are two ways to choose each of p, q, r, s, t giving $2^5 = 32$ choices. So there are 32 different values of m. Hence there are 32 cases where $m = n$.

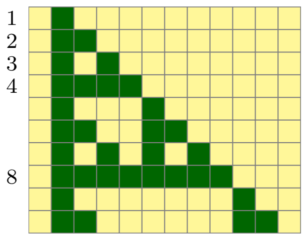
Subtracting these, there are $243 - 32 = 211$ pairs (m, n),

hence (211).

28. *Alternative 1*

Continue the pattern a bit more, using the observation that once we know two adjacent squares A, B in the same row, the square C below B is gold if A and B are the same colour and green if not.





Let $f(n)$ be the number of green tiles up to and including row n. Then $f(1) = 1$, $f(2) = 3$, $f(3) = 5$ and $f(4) = 9$. Since the pattern for $n = 8$ has 3 copies of the pattern for $n = 4$, we have that $f(8) = 27$.

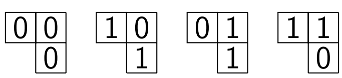
Continuing down, the green squares at the ends of the 9th and 10th row will develop into two copies of the $n = 8$ pattern, separated by a gold triangle. This continues up to row 16, so that $f(16) = 3 \times 27 = 81$. Also row 16 consists of a line of 16 green tiles in an otherwise gold row.

Then row 17 will have two green squares at distance 16, with the rest of the row being gold. Each of these two green squares will be the top square in two more copies of the overall pattern. However, this only continues up to row 20, which is also when the green squares reach the rightmost column. So the number of green squares up to and including the 20th row is $f(20) = f(16) + 2f(4) = 81 + 2 \times 9 = 99$,

hence (99).

Alternative 2

Number green squares '1' and gold squares '0', so the problem is to sum all the values in the grid. As in the solution above, two adjacent values determine a value below:



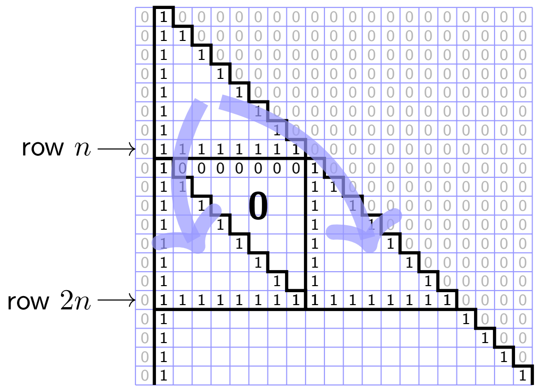
These rules lead to repeating structures in the pattern. In particular:

(a) The pattern of ones lies within the right isosceles triangle whose tip is the single 1 in the first row. So in our 21×20 grid, the pattern of ones only reaches the right side of the grid in the last row.

(b) When row n is $0111\ldots100\ldots$, with n ones, the next row is $0100\ldots010\ldots$ with $n-1$ zeroes between two ones. From here the rules apply as they did in rows $1, 2, 3, \ldots$. Consequently rows $(n+1)$ up to $2n$ consist of two copies of the pattern in rows 1 to n, as shown below. Here the left copy is shifted n rows down, and the right copy is shifted n rows down and n columns right.

(c) Following (b), row $2n$ is $0111\ldots1100\ldots$, with $2n$ ones. Consequently, each of rows $n = 1, 2, 4, \ldots, 2^k, \ldots$ has a line of n ones.

(d) Following (b), the sum of rows 1 to $2n$ is 3 times the sum of rows 1 to n.



For $n = 1, 2, \ldots$ let $f(n)$ be the sum of all rows up to and including row n. Then $f(1) = 1$ and due to (d) above, $f(2) = 3$, $f(2^2) = 3^2$, $f(2^3) = 3^3$, $\ldots$, $f(2^k) = 3^k$.

When n is not a power of 2, say $2^k < n < 2^{k+1}$, put $m = n - 2^k$ so that $n = 2^k + m$ and $0 < m < 2^k$. The first 2^k rows of the pattern sum to $f(2^k) = 3^k$. Then rows $2^k + 1$ up to $2^k + m$ contain two copies of the pattern in rows 1 to m, so these rows sum to $2f(m)$. Consequently, $f(n) = f(2^k + m) = f(2^k) + 2f(m) = 3^k + 2f(m)$. This allows us to find $f(n)$ for any positive integer n (assuming no width constraint).

In particular, for $n = 20$, decompose $20 = 2^4 + 2^2$ into powers of 2, then

$$f(4) = f(2^2) = 9$$
$$f(20) = f(2^4 + 2^2) = 3^4 + 2f(4) = 81 + 2 \times 9 = 99$$

That is, there are 99 green squares up to and including the 20th row,

hence (99).

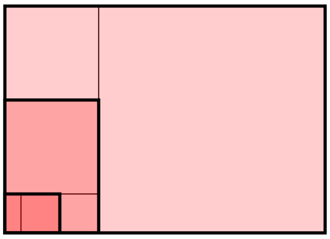
Note: Since the rules for zeroes and ones agree with modulo-2 addition, this version contains the modulo-2 values of the numbers in Pascal's triangle. Consequently it shows the pattern of odd and even numbers in Pascal's triangle.

29. *Alternative 1*

Let the rectangle after n steps have long side a_n metres and short side b_n metres, so that the square removed has side b_n. The perimeter will be $P = 4(b_0 + b_1 + b_2 + b_3 + \cdots)$ metres. We tabulate the first few cases keeping track of the ratio of the two sides, which we simplify by rationalising denominators:

n	a_n	b_n	a_n/b_n
0	$\sqrt{2} \approx 1.414$	1	$\sqrt{2}$
1	1	$\sqrt{2} - 1 \approx 0.414$	$\dfrac{1}{\sqrt{2}-1} = \sqrt{2} + 1$
2	$2 - \sqrt{2} \approx 0.586$	$\sqrt{2} - 1 \approx 0.414$	$\dfrac{2-\sqrt{2}}{\sqrt{2}-1} = \sqrt{2}$
3	$\sqrt{2} - 1 \approx 0.414$	$3 - 2\sqrt{2} \approx 0.172$	$\dfrac{\sqrt{2}-1}{3-2\sqrt{2}} = \sqrt{2} + 1$

Since the ratio of sides has repeated, the process continues in the same pattern, but scaled down for each cycle:



The shaded rectangles here are those with even n, and are all in the ratio $\sqrt{2}$ to 1. From one shaded rectangle to the next, the length of the sides is scaled down by a factor $\sqrt{2}-1$. Consequently for even $n = 2m$, $b_n = (\sqrt{2}-1)^m$, and for odd n, $a_n = b_{n-1}$.

Summing b_n for even n, we have an infinite geometric series:

$$\begin{aligned} b_0 + b_2 + b_4 + b_6 + \cdots &= 1 + (\sqrt{2}-1) + (\sqrt{2}-1)^2 + (\sqrt{2}-1)^3 + \cdots \\ &= \frac{1}{1-(\sqrt{2}-1)} \\ &= \frac{1}{2-\sqrt{2}} = 1 + \frac{\sqrt{2}}{2} \end{aligned}$$

The odd terms are the same, but without the first term, which is $b_0 = 1$:

$$b_1 + b_3 + b_5 + \cdots = \frac{\sqrt{2}}{2}$$

So the total perimeter is then 4 times these two sums:

$$\begin{aligned} P = 4\left(1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) &= 4\sqrt{2} + 4 \\ &\approx 4 \times 1.414 + 4 = 9.656 \end{aligned}$$

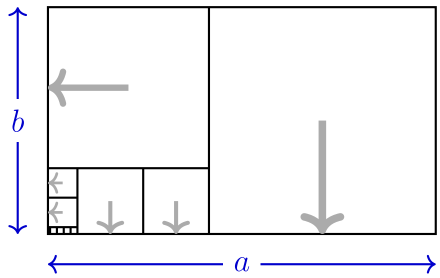
This suggests 966 as the answer. To confirm, we have $1.414 < \sqrt{2} < 1.415$ so that $9.656 < P < 9.66$. Then $P = 966\,\text{cm}$, to the nearest centimetre,

hence (966).

Alternative 2

Consider the more general problem of an $a \times b$ rectangle where $\frac{a}{b}$ is irrational. We will first find the sum of the series where each term is the side of one of the squares.

We can assume that the cuts always leave the remaining rectangle in the lower-left corner. Then each square either has one side on the bottom of the rectangle or on the left side of the rectangle. No square has both.

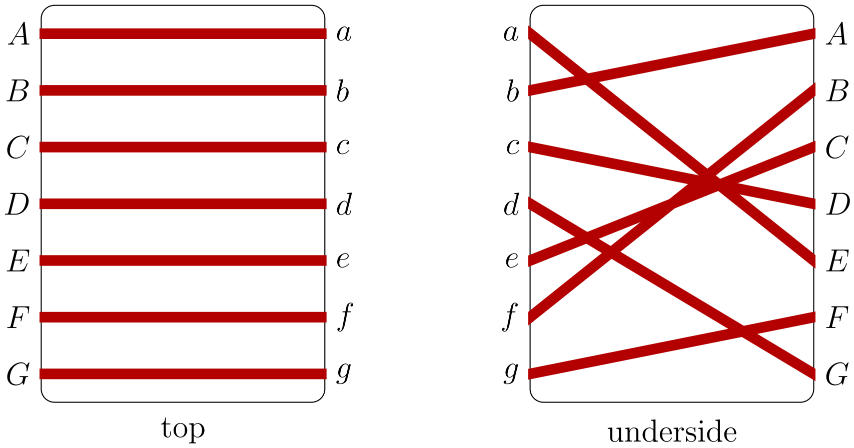


Consequently, a is the sum of the widths of all the squares along the bottom and b is the sum of the heights (hence the widths) of all the squares along the left side. Then $a + b$ is the sum of the widths of all the squares and $4(a + b)$ is the sum of the perimeters of all the squares.

For a $1 \times \sqrt{2}$ metre rectangle, $4(a + b) = 4(1 + \sqrt{2}) = 9.66\,\text{m}$ to the nearest $0.01\,\text{m}$,

hence (966).

30. Label the ends of the seven stripes by $A, B, C, \ldots, G$ and $a, b, c, \ldots, g$, as in the example shown below. On the top of the deck, the elastic band always joins $A \to a$, $B \to b$, $C \to c$, $\ldots$, $G \to g$.



On the underside of the deck in the example above, the elastic band joins $a \to E$, $b \to A$, $c \to D$ and so on. Combining the joins for the top and underside, we can describe the pattern by the sequence

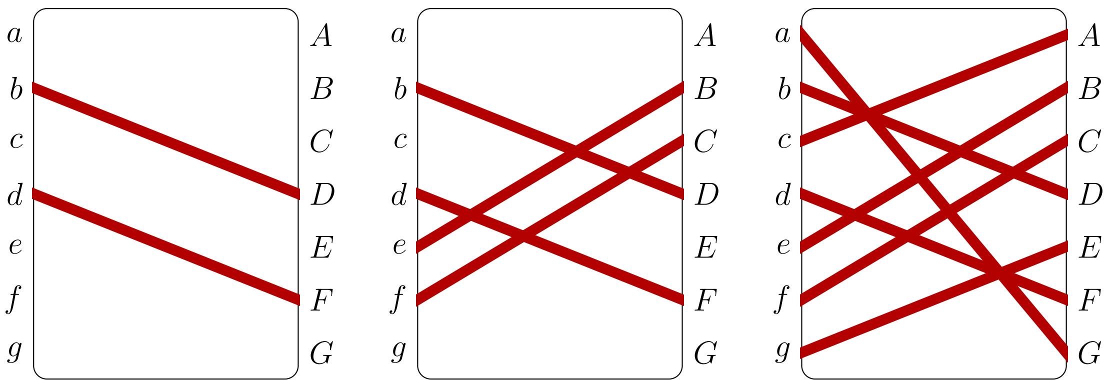
$$A \to a \to E \to e \to C \to c \to D \to d \to G \to g \to F \to f \to B \to b \to A$$

By ignoring the lowercase labels, since they always match the preceding uppercase labels, we can simplify the sequence of the example shown:

$$A \to E \to C \to D \to G \to F \to B \to A$$

Since the elastic band forms a single continuous loop, every label occurs exactly once in any such sequence, with the exception of A, which is repeated at the end to close the loop. Thus every pattern corresponds to an ordering of the six labels $B, C, \ldots, G$ in a sequence starting and ending with A. There are $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ such orderings, but we now need to account for those sequences representing patterns that are rotations of each other.

To this end, we determine the number of patterns which are equal to their own rotation, that is, which are rotationally symmetrical. Consider first the stripe $D \to d$. On the underside, d can join to any of the six labels excluding D, but any such choice automatically determines the join ending at D due to rotational symmetry; for example, if $d \to F$ then $b \to D$, as shown on the left below. Following the most recent stripe back to the left, there are now four choices for the next join; for example, tracing F back to f, we cannot join to B since this prematurely closes the loop, so the only options are A, C, E or G. As before, this then determines another join due to symmetry; for example, if $f \to C$, then $e \to B$, as shown in the middle diagram.



$d \to F,\ b \to D$ $\qquad$ $f \to C,\ e \to B$ $\qquad$ $c \to A,\ g \to E,\ a \to G$

Similarly, the most recent stripe back to the left gives a further 2 choices for the next join, $c \to A$ say, and the remaining two joins are then determined. Hence the number of choices at each stage is 6, then 4, then 2, giving a total of $6 \times 4 \times 2 = 48$ patterns, which are rotationally symmetrical.

Each of the 48 rotationally symmetrical patterns counts exactly once towards the total. The remaining $720 - 48 = 672$ sequences occur in pairs representing patterns which are rotations of each other. Since only one pattern per pair should count, there are an additional $672 \div 2 = 336$ patterns and therefore the total number is $336 + 48 = 384$,

hence (384).

Answer Key

Question	Middle Primary	Upper Primary	Junior	Intermediate	Senior
1	B	E	C	D	E
2	C	E	B	C	D
3	E	E	B	A	C
4	D	A	B	D	A
5	C	A	C	B	B
6	D	D	E	B	C
7	E	A	B	E	B
8	D	E	A	B	B
9	B	D	C	B	C
10	A	D	E	D	C
11	E	E	B	C	D
12	E	D	A	D	B
13	E	D	D	C	A
14	B	A	D	A	C
15	B	B	D	C	A
16	A	C	B	E	D
17	C	C	C	A	B
18	B	C	E	C	E
19	A	C	A	E	E
20	C	A	C	E	E
21	C	E	B	E	C
22	B	B	B	C	E
23	D	A	C	D	E
24	A	B	C	B	C
25	E	D	A	E	D
26	198	666	356	71	245
27	200	72	84	36	211
28	385	45	250	55	99
29	602	17	40	93	966
30	17	117	232	792	384

AMC 2021 SOLUTIONS